# Equilibrium of Compression-only Structures Made of Convex Polyhedra

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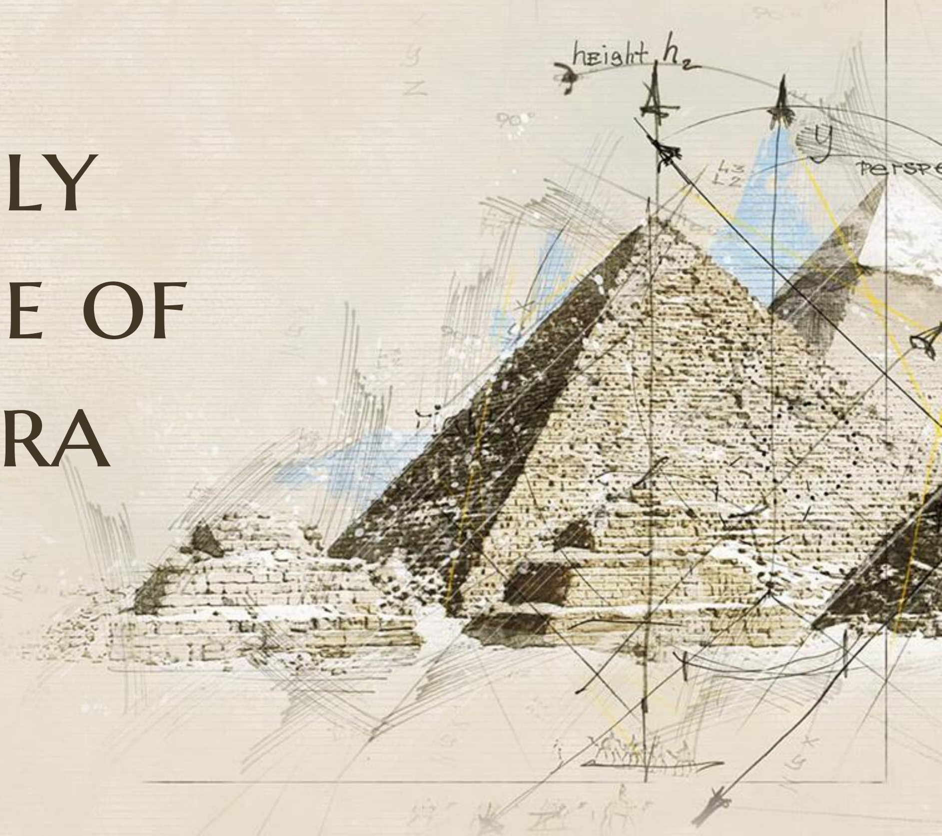
Advisor: Christoph Hoffmann



# Let's talk about…

**1. Compression Structures**

Basic definition and some examples.

**2. Equilibrium Analysis**

Reasoning about equilibrium and forces.

**3. Assemblies Made of Convex Polyhedra**

Counterintuitive structures made possible.

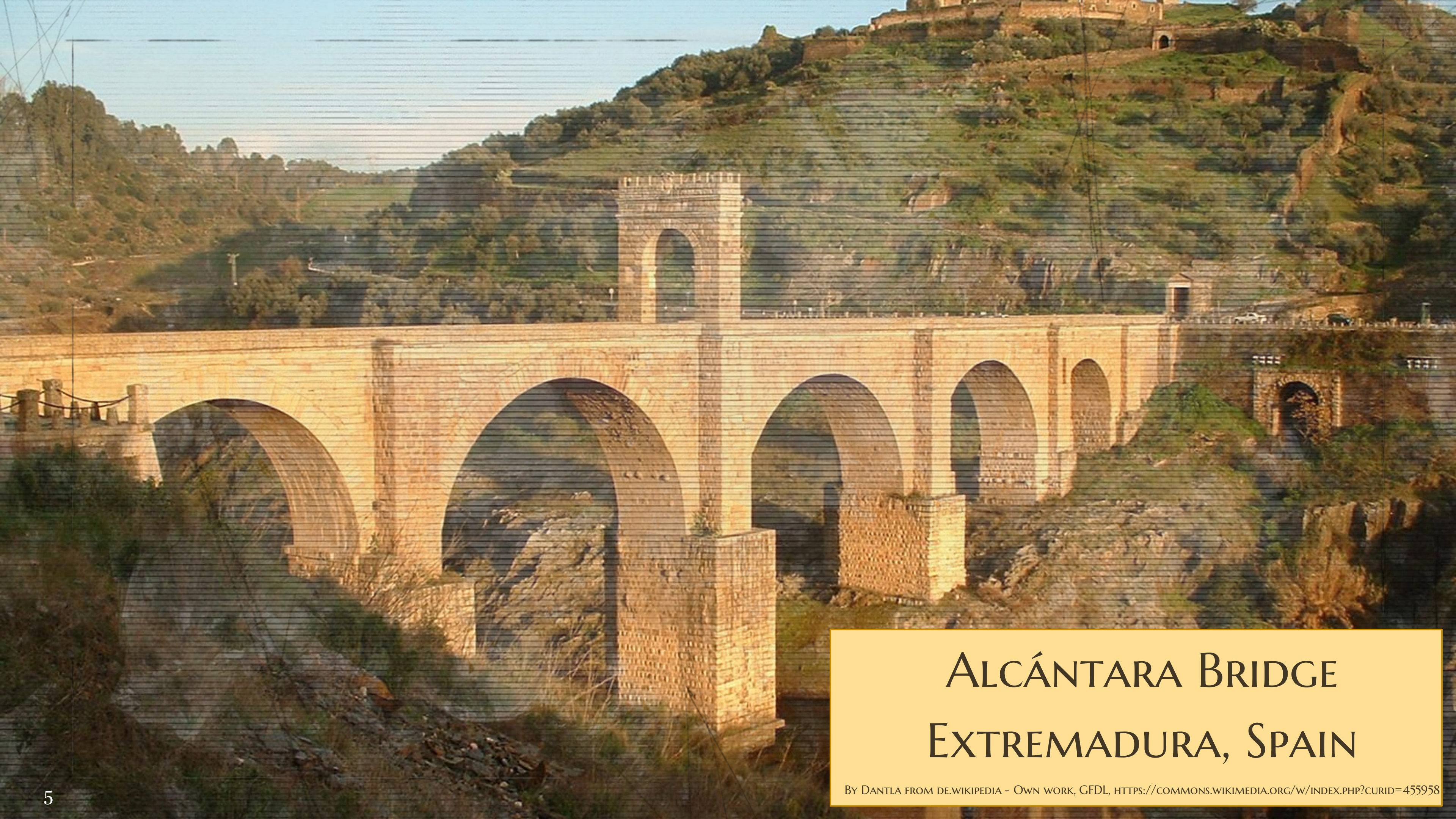
# 1.

# Compression Structures

What are they and why they matter

# Compression Structure

- Compressive loads applied along the structure.
- The compression load applied to the cross section of structure produces stress.
- Best when using rigid materials.



# Alcántara Bridge
# Extremadura, Spain

# Igloo at Night, Vermillion Lakes, Banff National Park, Alber Bridge

By Jochi, https://wall.alphacoders.com/big.php?i=437732

# Droneport Prototype
# Venice, Italy

Block Research Group (BRG), © Nigel Young and The Norman Foster Foundation
https://block.arch.ethz.ch/brg/project/venice-biennale-2016_droneport

# The Red Line project

Block Research Group (BRG), © Foster + Partners
https://block.arch.ethz.ch/brg/project/venice-biennale-2016_droneport

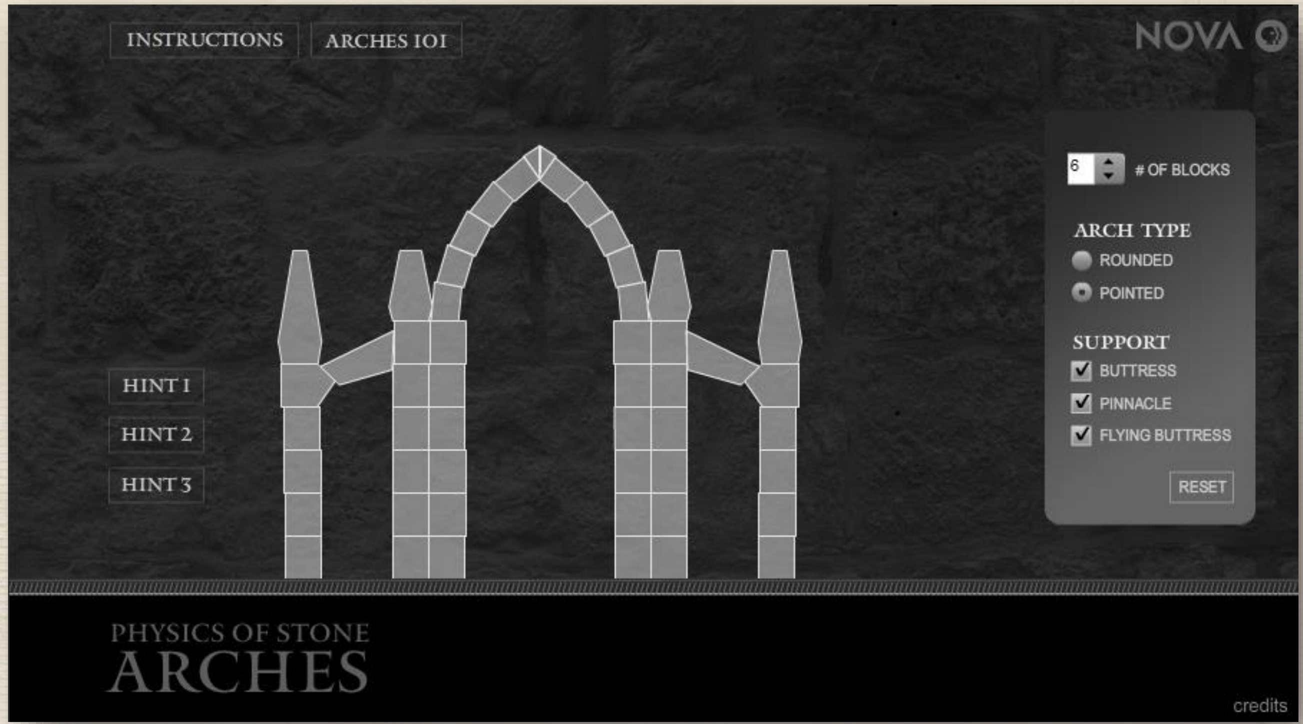
# Sagrada Família
# Barcelona, Spain

https://wallpapercave.com/w/wp3766514

Von Rüdiger Marmulla - Eigenes Werk, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=4316670

# Pringles Ring

http://www.blazenfluff.com/one-pringle-to-rule-them-all-how-to-make-a-stacked-ring-of-pringles/5672

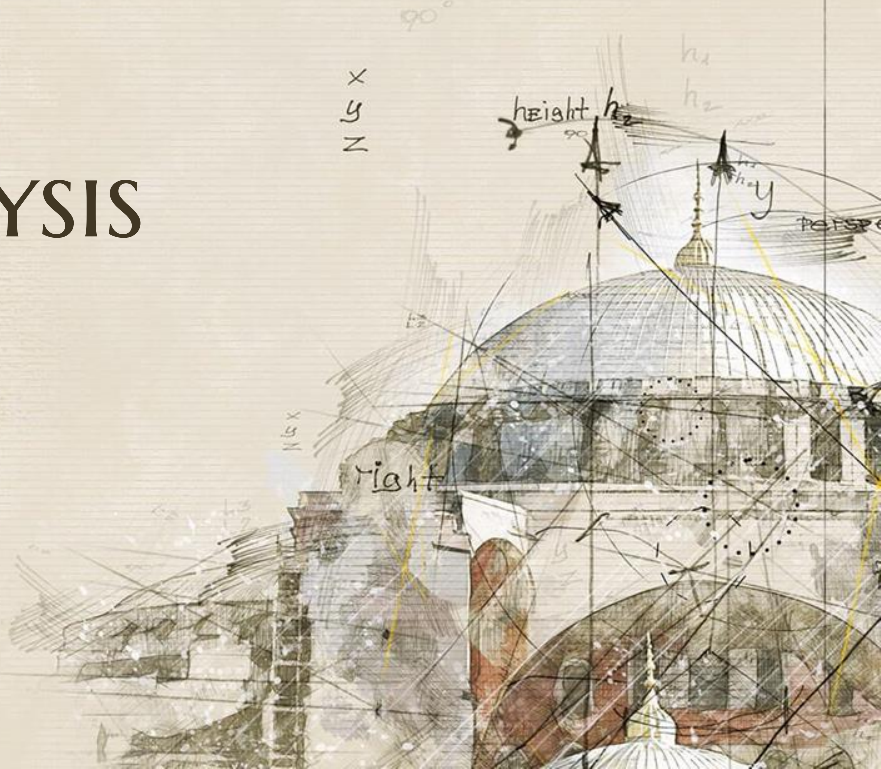



# Physics of Stone Arches (Demo)

https://www.pbs.org/wgbh/nova/physics/arch-physics.html

# 2.
# Equilibrium Analysis

A bit of formality



**R. K. Livesley**
*Cambridge*

**Emily Whiting**
*MIT*

**Me**
*Purdue*

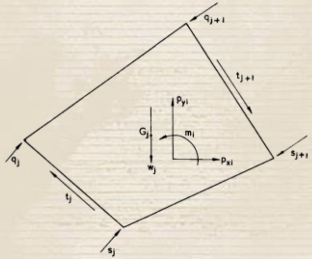
*1978* — *No forces? Then no possible*

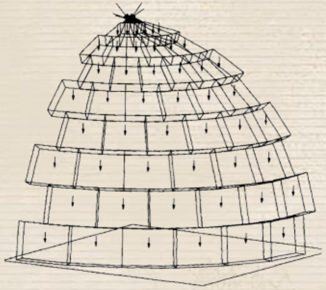
*1992* — *Even in 3D!*

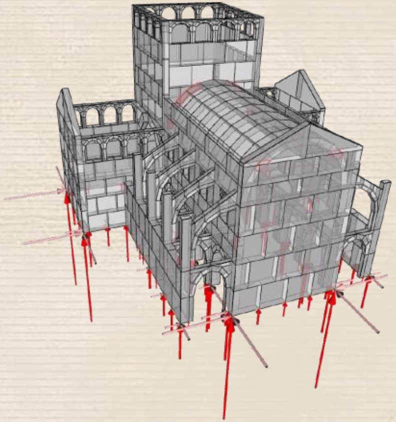
*...*

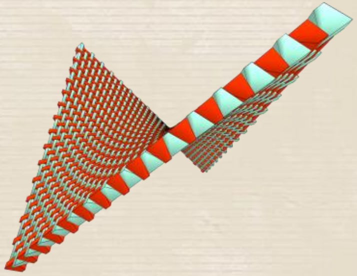
*2009* — *Let's relax*

*2019* — *Please, don't move!*

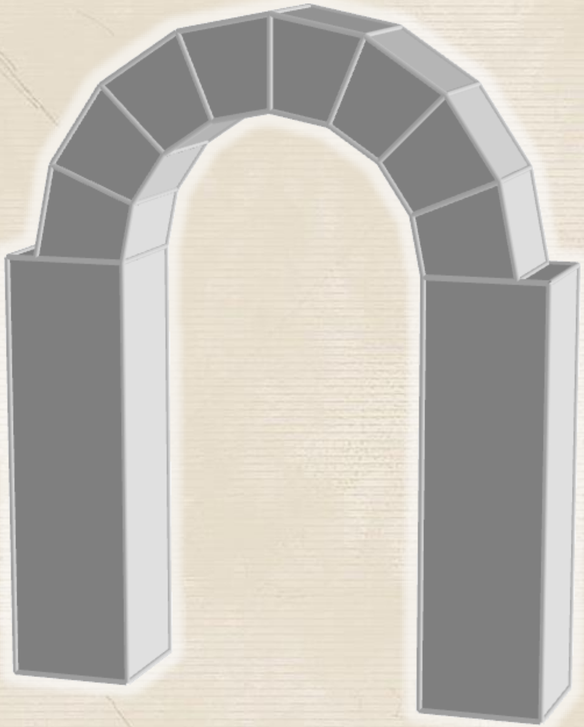


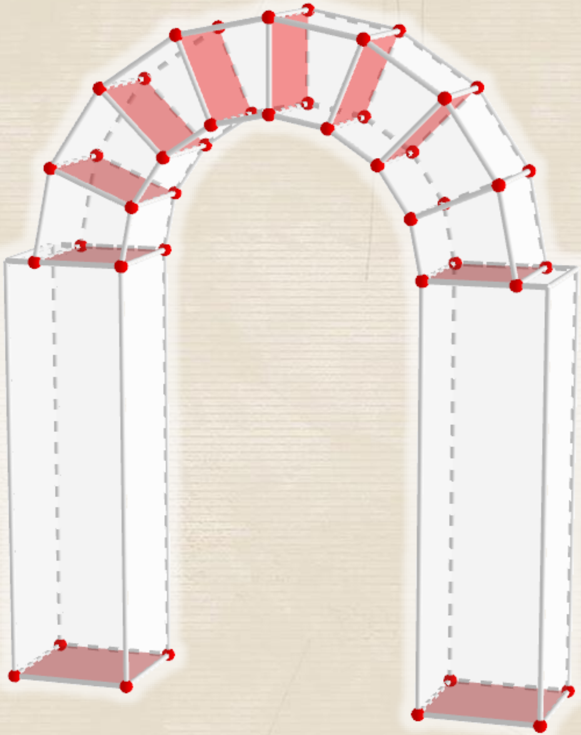






Livesley, R. K. "Limit *Analysis* of Structures Formed from Rigid Blocks." *International Journal for Numerical Methods in Engineering* 12, no. 12 (1978): 1853–71.
Livesley, R. K. "A Computational Model for the Limit Analysis of Three-Dimensional Masonry Structures." *Meccanica* 27, no. 3 (1992): 161–72.
Whiting, Emily, John Ochsendorf, Frédo Durand, "Procedural Modeling of Structurally-Sound Masonry Buildings." In *ACM Transactions on Graphics (TOG)*, 28:112. ACM, 2009.





Forward Statics Problem

# 1. Density

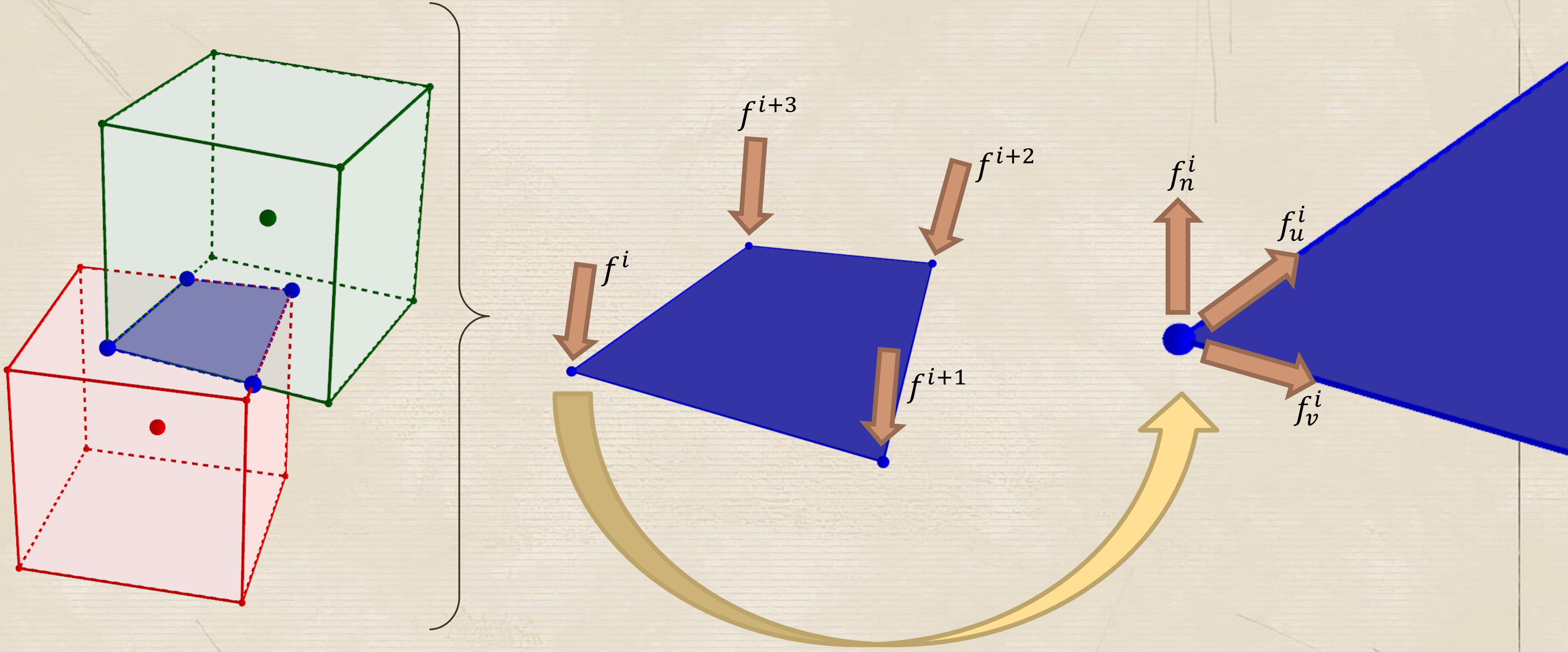
It is uniform for all blocks

# 2. Friction

It exists between blocks

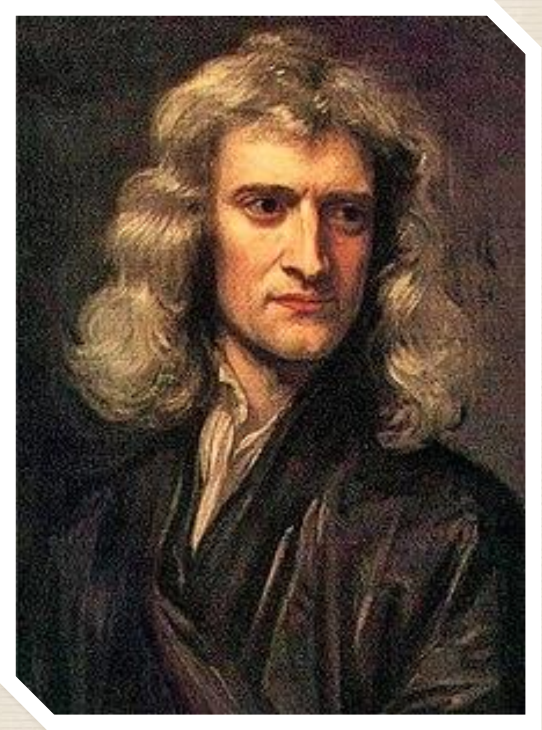
# 3. Gravity

It pulls everything down




# Interface Polygons & Contact Forces

# Newton's First Law of Motion

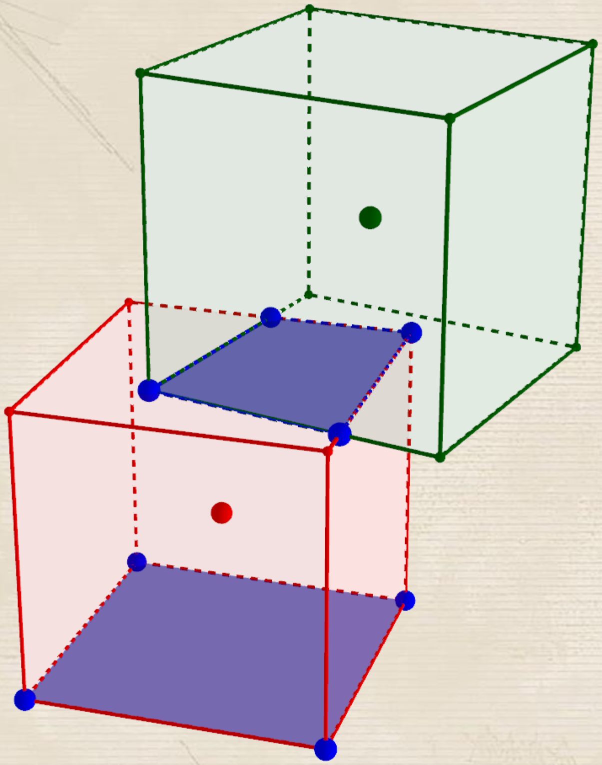


By After Godfrey Kneller - http://www.newton.cam.ac.uk/art/portrait.html, Public Domain

https://commons.wikimedia.org/w/index.php?curid=37337

*"Every body persists in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed."*

$$\sum \boldsymbol{F} = 0$$



Equations per block

Net force and net torque per block:

$\sum F = 0$ and $\sum T = 0$

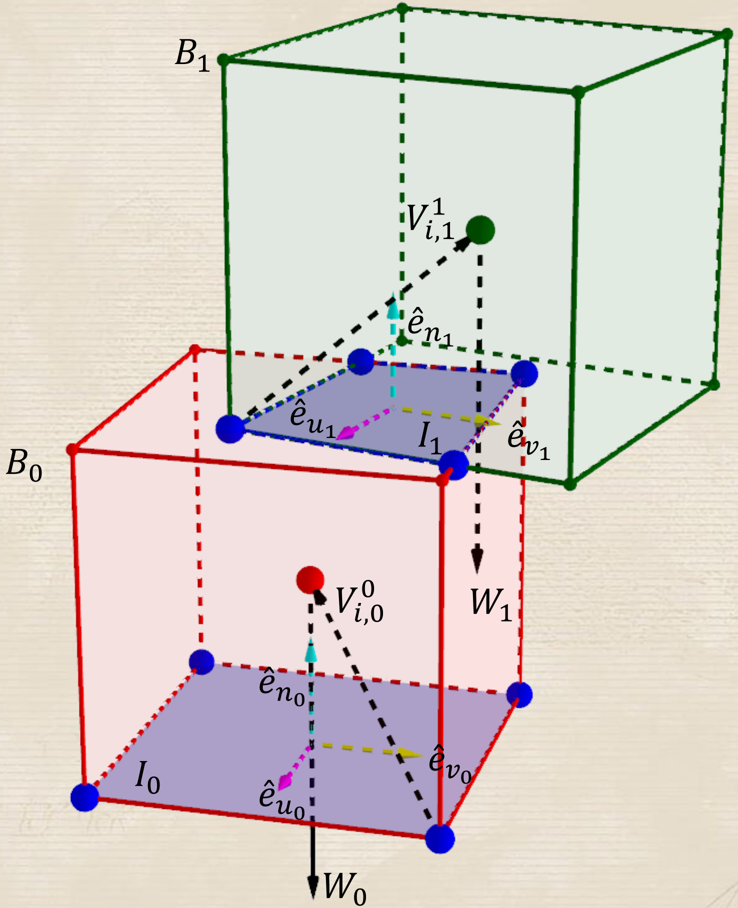
Six equations per block:

$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0$

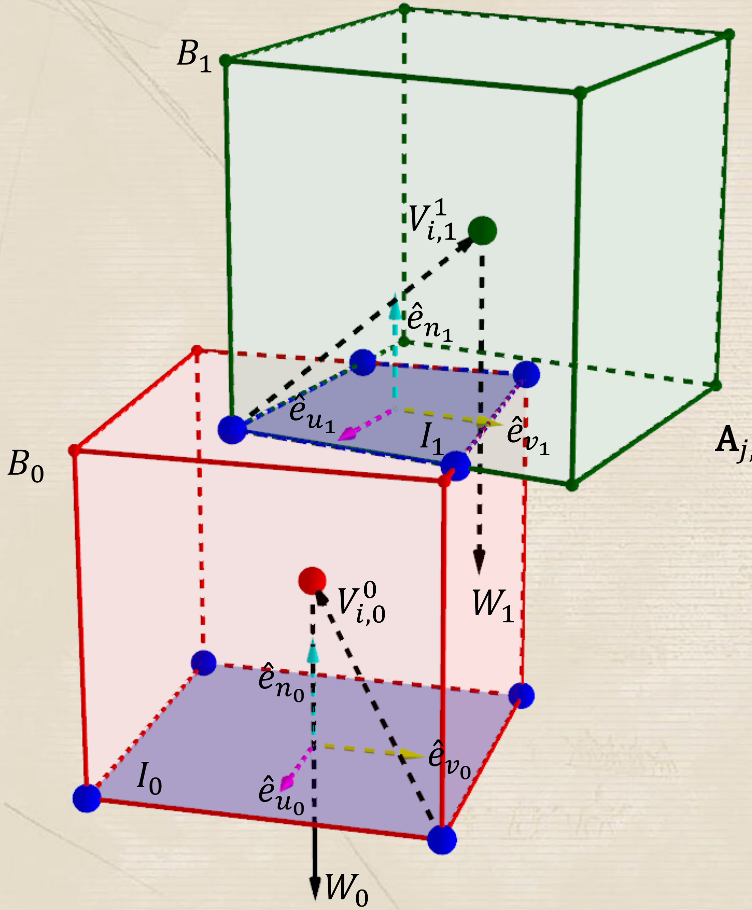
$\sum T_x = 0 \quad \sum T_y = 0 \quad \sum T_z = 0$

Each interface contributes with forces and torques on its incident blocks.

Interactions with the ground are considered as interfaces too.




*Elements between interface $I_1$ and block $B_0$ are left as an exercise to the reader.




## Equations for force contribution from interface $I_k$ on block $B_j$:

$$\mathbf{A}_{j,k} r_k = \begin{bmatrix} \mathbf{F}_k & \mathbf{F}_k & \cdots \\ \mathbf{T}_{i,j,k} & \mathbf{T}_{i+1,j,k} & \cdots \end{bmatrix} \begin{bmatrix} \mathbf{f}^i \\ \mathbf{f}^{i+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathrm{a}_{k_x} & \mathrm{a}_{k_x} & \cdots \\ \mathrm{a}_{k_y} & \mathrm{a}_{k_y} & \cdots \\ \mathrm{a}_{k_z} & \mathrm{a}_{k_z} & \cdots \\ \mathrm{b}_{i,j,k_x} & \mathrm{b}_{i+1,j,k_x} & \cdots \\ \mathrm{b}_{i,j,k_y} & \mathrm{b}_{i+1,j,k_y} & \cdots \\ \mathrm{b}_{i,j,k_z} & \mathrm{b}_{i+1,j,k_z} & \cdots \end{bmatrix} \begin{bmatrix} \mathbf{f}^i \\ \mathbf{f}^{i+1} \\ \vdots \end{bmatrix}$$

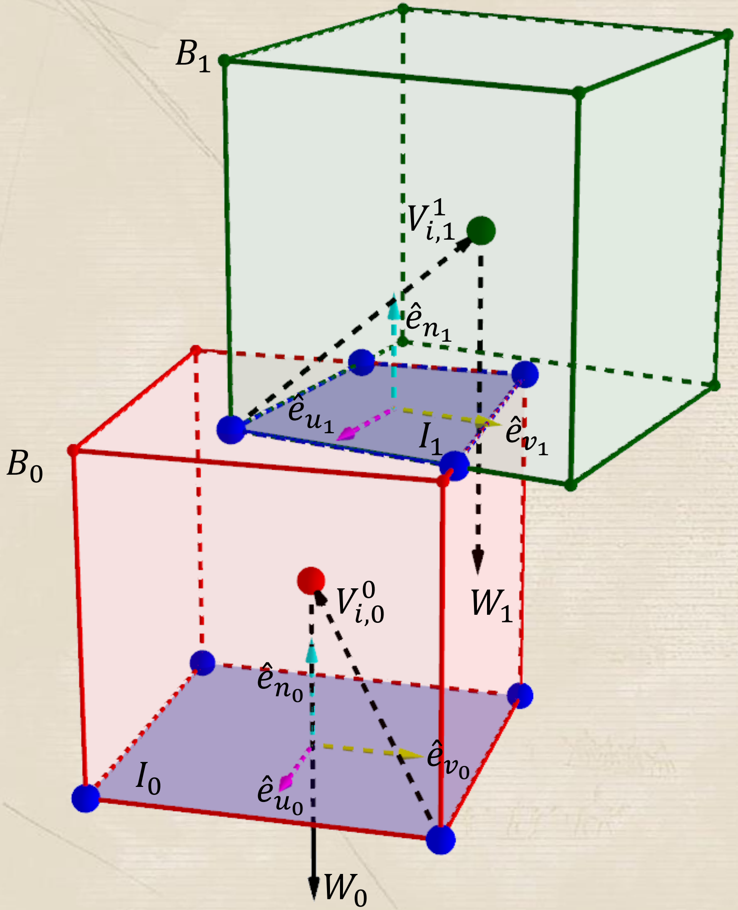
$$\mathbf{f}^i = [f_n^i \quad f_u^i \quad f_v^i]^T$$

$$\mathrm{a}_{k_x} = [\hat{e}_{n_{k_x}} \quad \hat{e}_{u_{k_x}} \quad \hat{e}_{v_{k_x}}]$$

$$\mathrm{b}_{i,j,k_x} = \left[\left(\hat{e}_{n_k} \times V_{i,j}^k\right)_x \quad \left(\hat{e}_{u_k} \times V_{i,j}^k\right)_x \quad \left(\hat{e}_{v_k} \times V_{i,j}^k\right)_x\right]$$



*Flip $a_k$ when $\hat{e}_{n_k}$ faces an opposite direction with respective of the centroid of a block.




Equation for all interactions between interfaces and blocks:

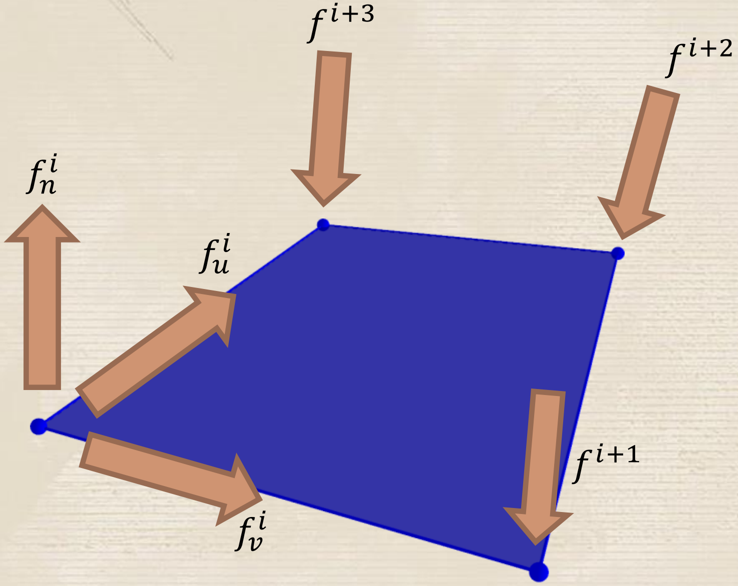
$$\begin{bmatrix} \mathbf{A}_{0,0} & \mathbf{A}_{0,1} & & \\ & \ddots & & \\ & & \mathbf{A}_{n-1,n-1} & \mathbf{A}_{n-1,n} \end{bmatrix} \begin{bmatrix} \mathrm{r}_0 \\ \vdots \\ \mathrm{r}_n \end{bmatrix} + \begin{bmatrix} \mathrm{w}_0 \\ \vdots \\ \mathrm{w}_{n-1} \end{bmatrix} = 0$$

$$\mathbf{A}_{eq} \cdot \mathbf{f} + \mathbf{w} = 0$$

$A_{j,k}$: Submatrix of coefficients for net contributions from interface $k$ acting on block $j$.

$\mathrm{w}_j$: $6 \times 1$ vector containing the 3D weight and net torque for block $j$.

$\mathrm{r}_k$: Unknown force vectors for vertices $i$ on interface $k$.




**Compression Constraint:**

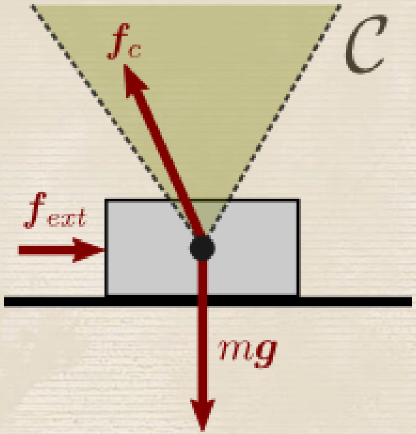
$$f_n^i \geq 0, \forall i \in \text{interface vertices}$$

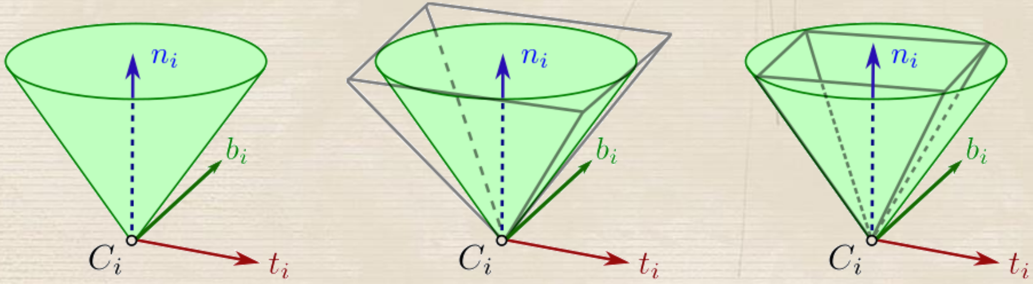
**Linearized Friction Constraints:**

$$|f_u^i|, |f_v^i| \leq \mu\, f_n^i, \forall i \in \text{interface vertices}$$

$$\mathbf{A}_{fr} \cdot \mathbf{f} \leq 0$$

Force Constraints




$$|f_u^i|, |f_v^i| \leq \mu \, f_n^i, \forall i \in \text{interface vertices}$$

$$|f_u^i|, |f_v^i| \leq \frac{\mu \, f_n^i}{\sqrt{2}}, \forall i \text{ (conservative)}$$

8-sided pyramids are also possible
(requires more calculations)

Why linearized friction constraints?



Friction Cones. https://scaron.info/teaching/friction-cones.html

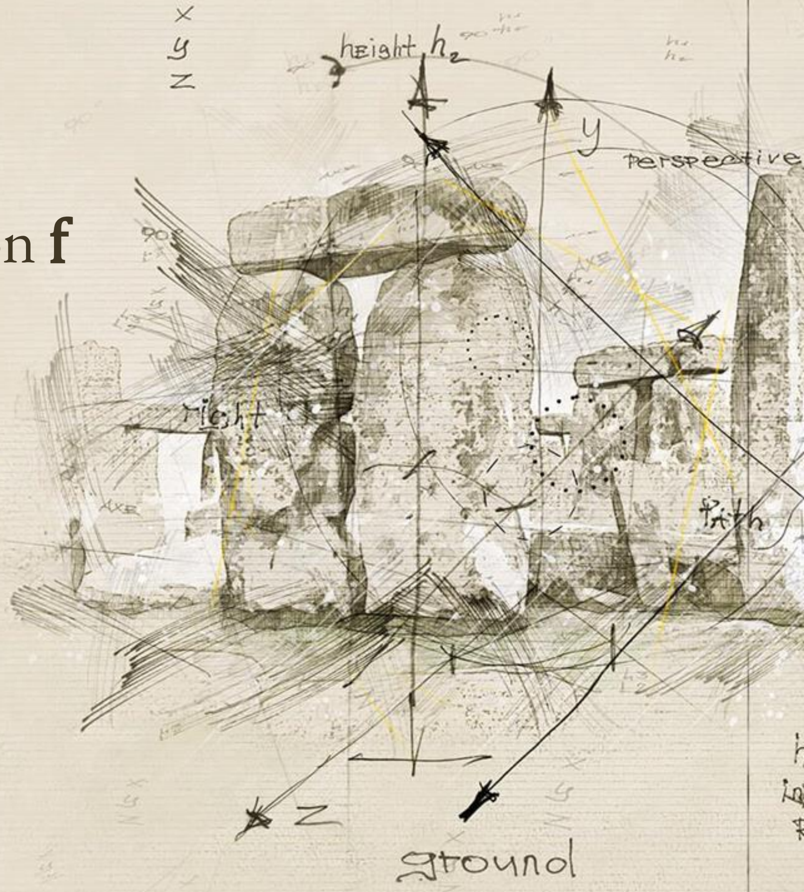
# Remember Livesley?

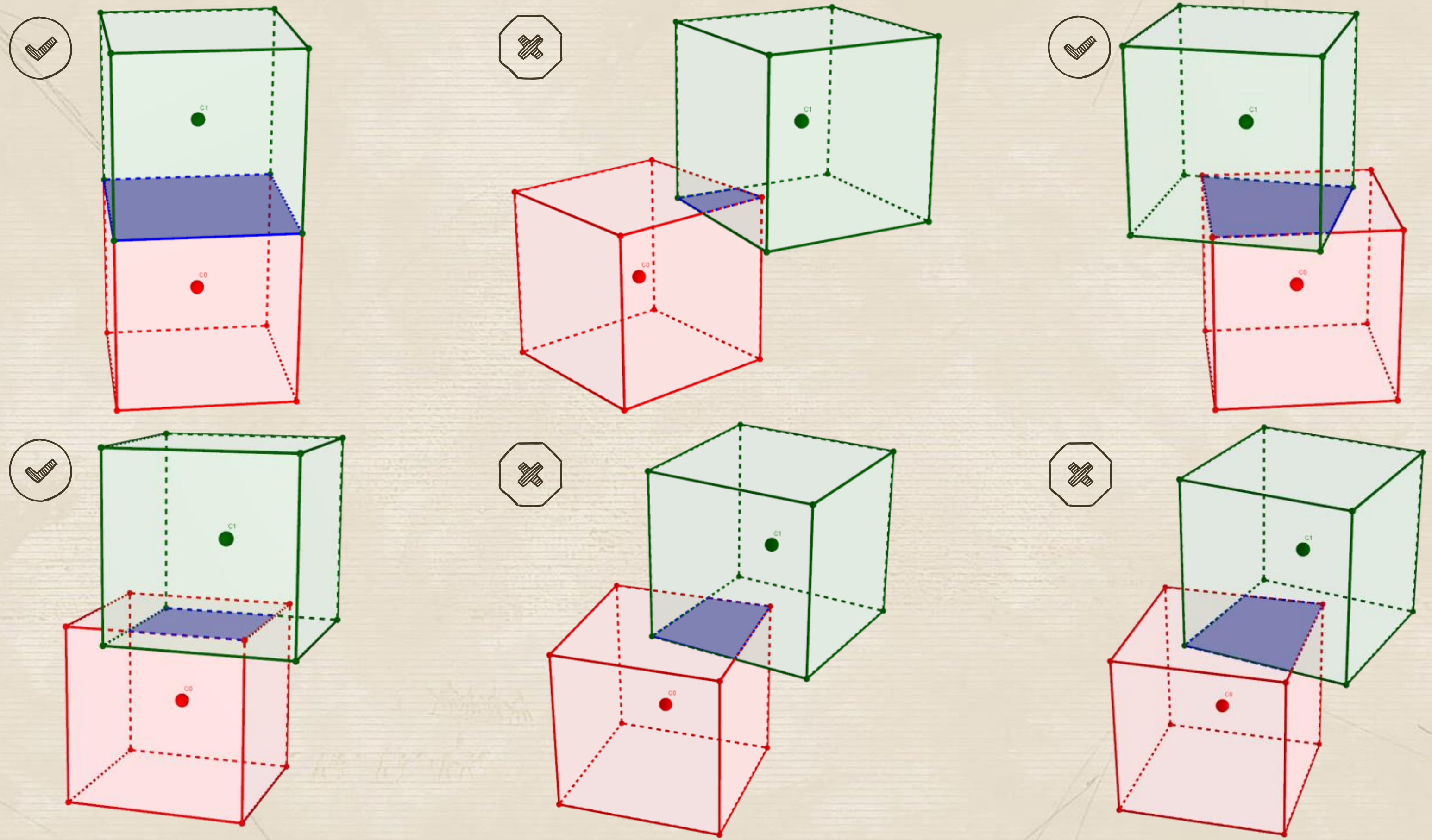
Structure in equilibrium if a force solution **f** exists that satisfies:

$$\mathbf{A}_{eq} \cdot \mathbf{f} + \mathbf{w} = 0$$

$$\mathbf{A}_{fr} \cdot \mathbf{f} \leq 0$$

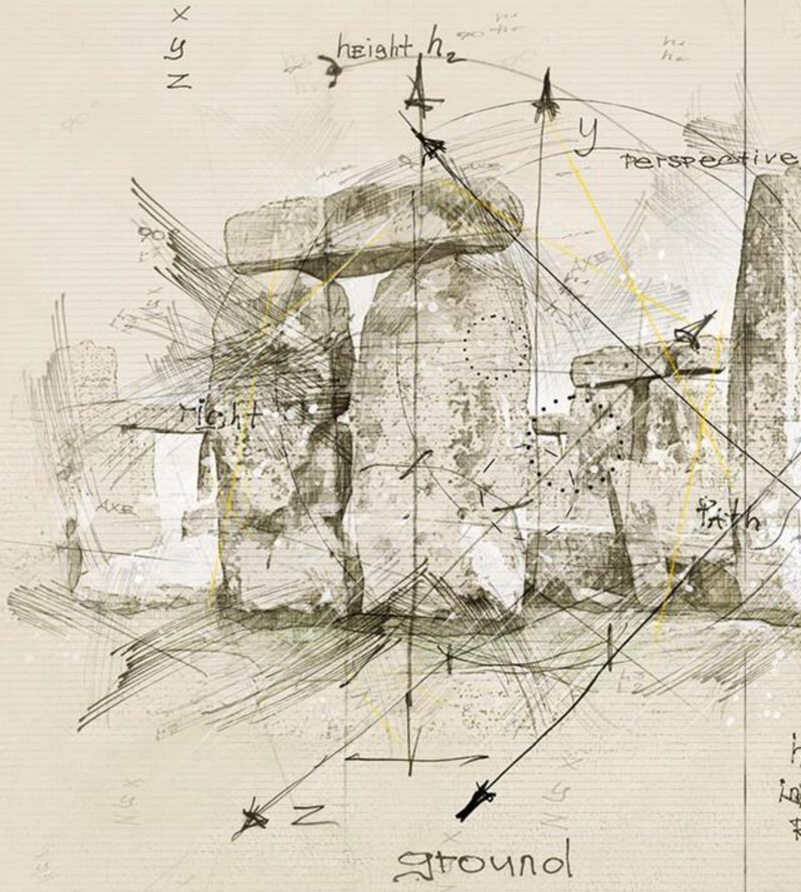
$$f_n^i \geq 0, \forall i \in \text{interface vertices}$$




C1
C0
C1
C0
C1
C0
C1
C0
C1
C0
C1
C0

# Whiting said relax!

Use glue!

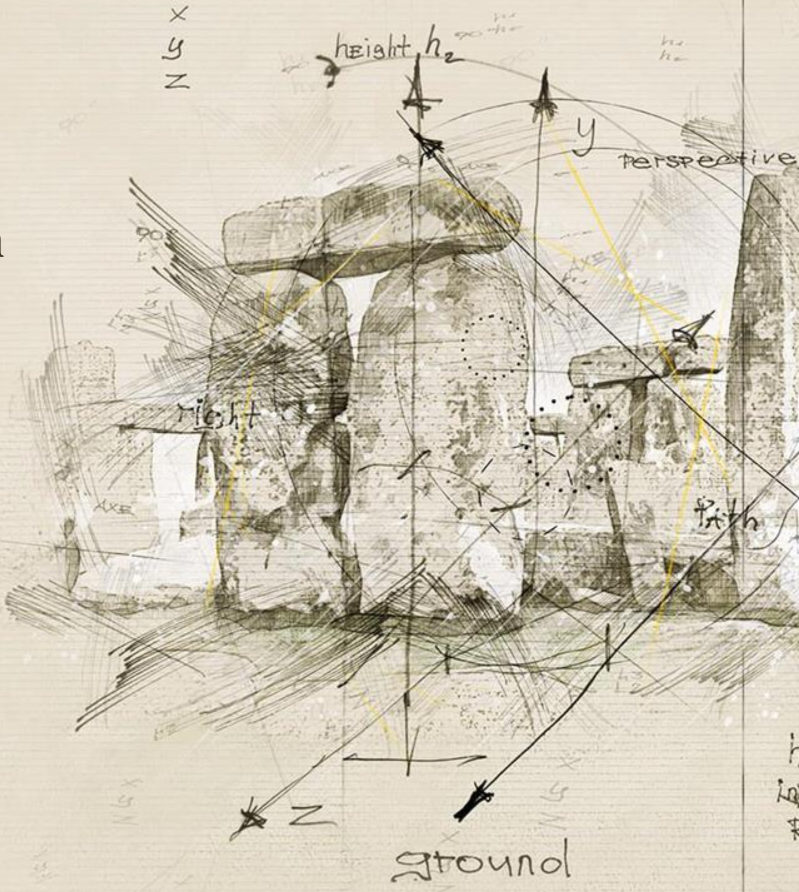



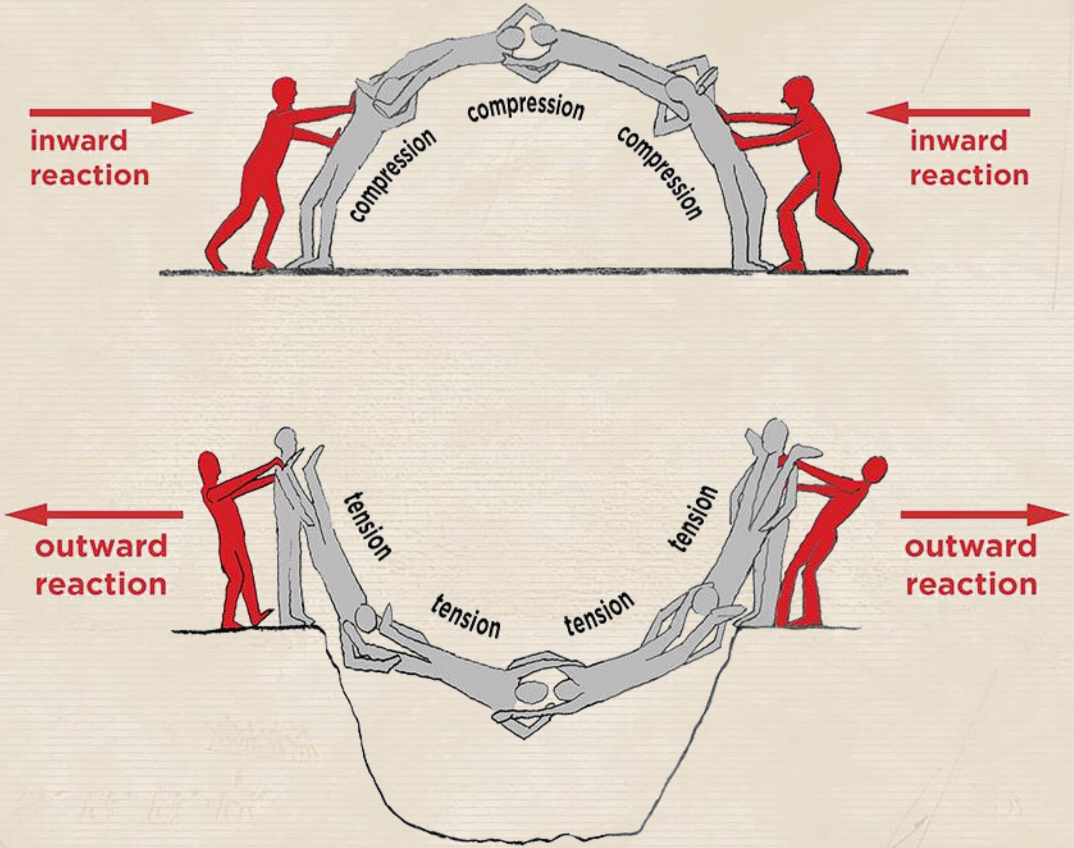
# Whiting said relax!

Decompose $f_n^i$ into compression and tension components…

$$f_n^i = f_n^{i+} - f_n^{i-}$$

$$f_n^{i+}, f_n^{i-} \geq 0$$







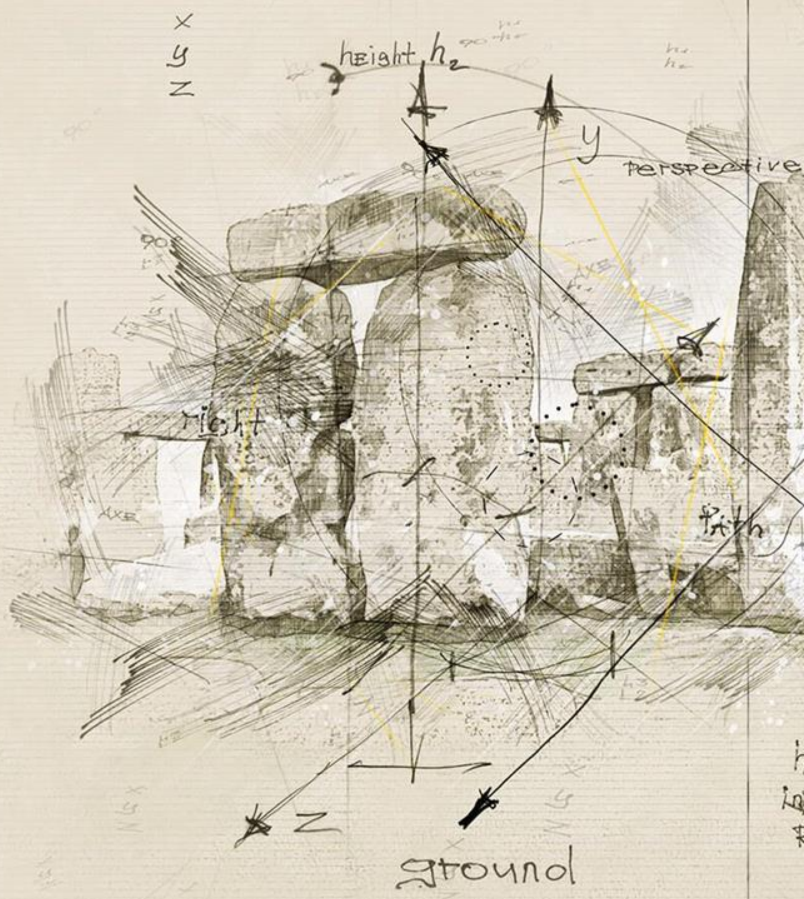
http://goldengate.org/_print.php?_url=http%3A%2F%2Fgoldengate.org%2Fexhibits%2Ffort-point-arch.php

# Whiting said relax!

... but penalize tensions:

$$\min_{\mathbf{f}} \sum_{i=0}^{n} \left(f_n^{i-}\right)^2$$

$$\mathbf{A}_{eq} \cdot \mathbf{f} + \mathbf{w} = 0$$
$$\mathbf{A}_{fr} \cdot \mathbf{f} \leq 0$$
$$f_n^{i+}, f_n^{i-} \geq 0, \forall i$$





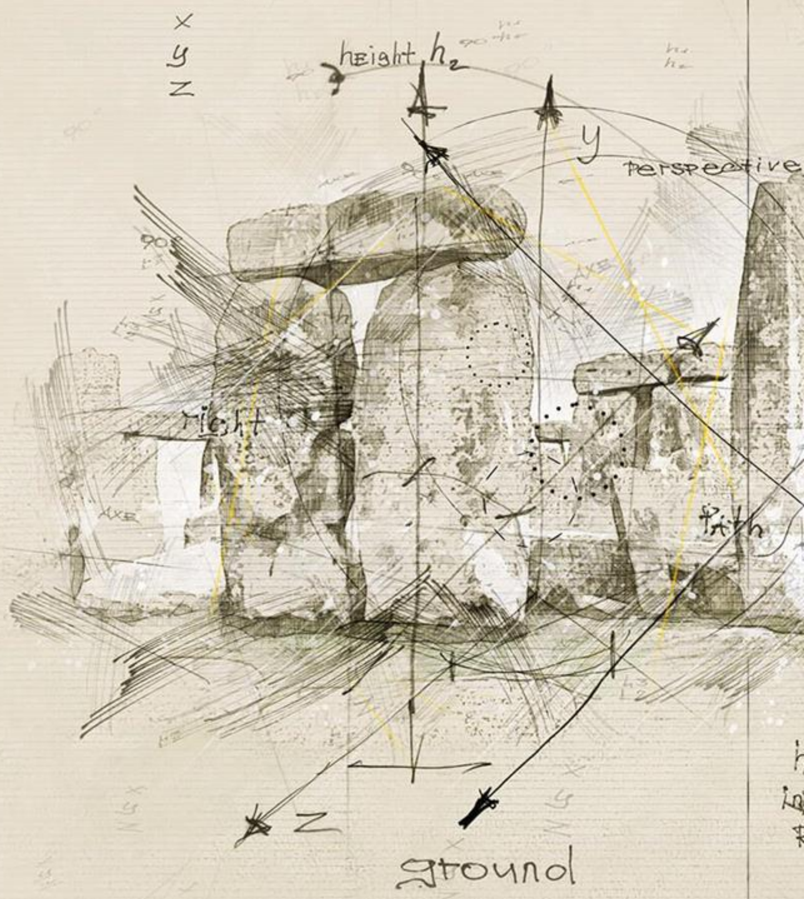

*Update $\mathbf{A}_{eq}$, $\mathbf{A}_{fr}$ and $\mathbf{f}$ for handling compression and tension components.

# Whiting said relax!

Quadratic program for all forces:

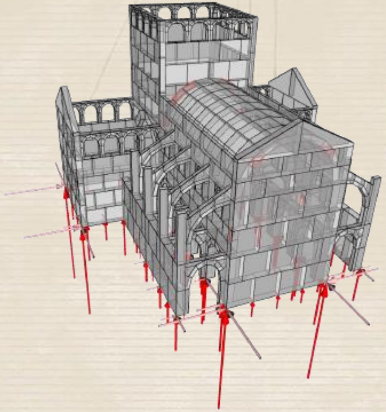
$$g(\mathbf{f}) = \min_{\mathbf{f}} \frac{1}{2}\mathbf{f}^T \mathrm{H}\mathbf{f}$$

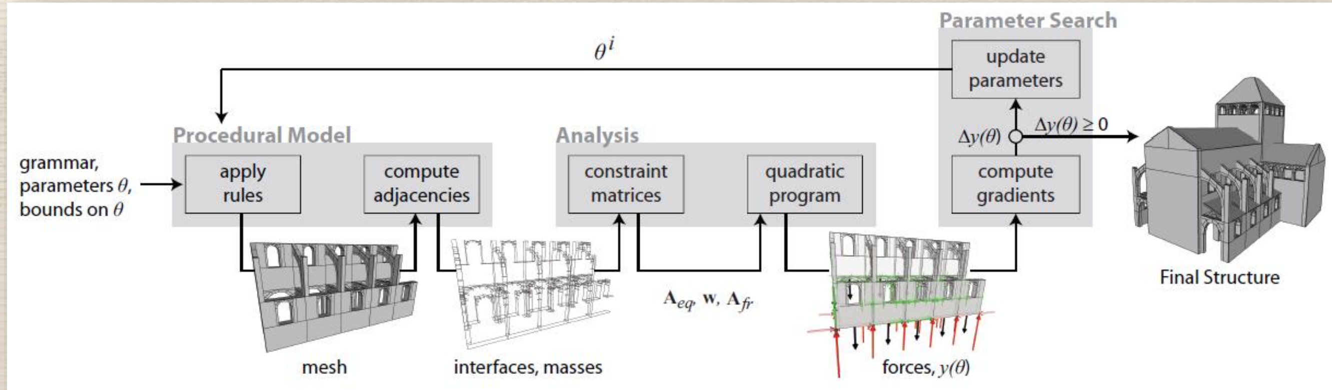
$$\mathbf{A}_{eq} \cdot \mathbf{f} + \mathbf{w} = 0$$
$$\mathbf{A}_{fr} \cdot \mathbf{f} \leq 0$$
$$f_n^{i+}, f_n^{i-} \geq 0, \forall i$$



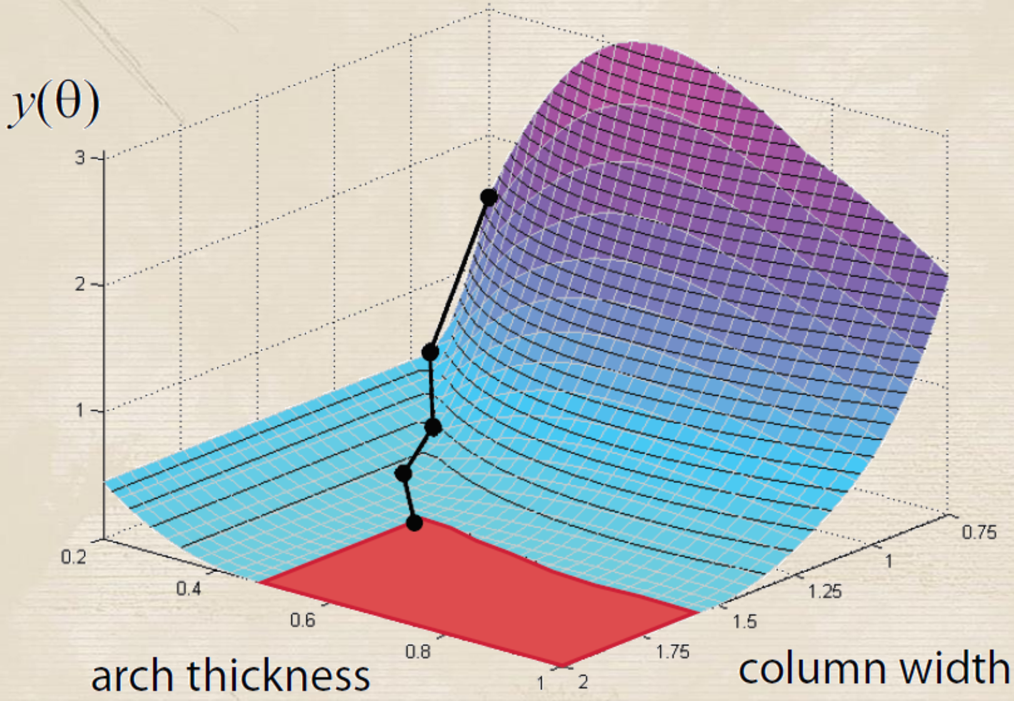


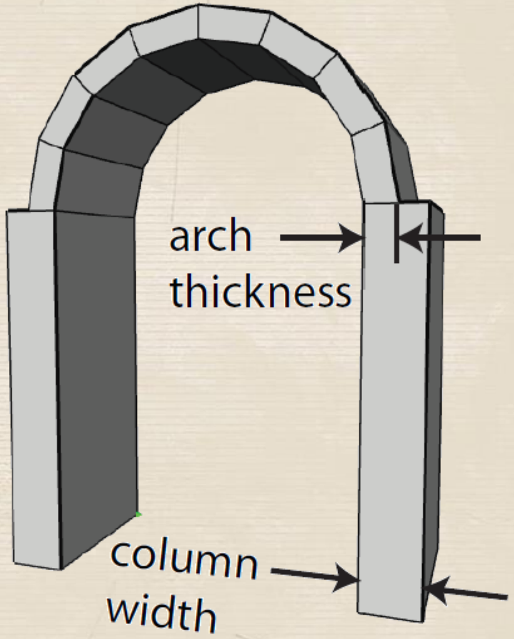
*Solution exists if H is positive semidefinite.






Whiting, Emily, John Ochsendorf, Frédo Durand, Emily Whiting, John Ochsendorf, and Frédo Durand. “Procedural Modeling of Structurally-Sound Masonry Buildings.” In *ACM Transactions on Graphics (TOG)*, 28:112. ACM, 2009.



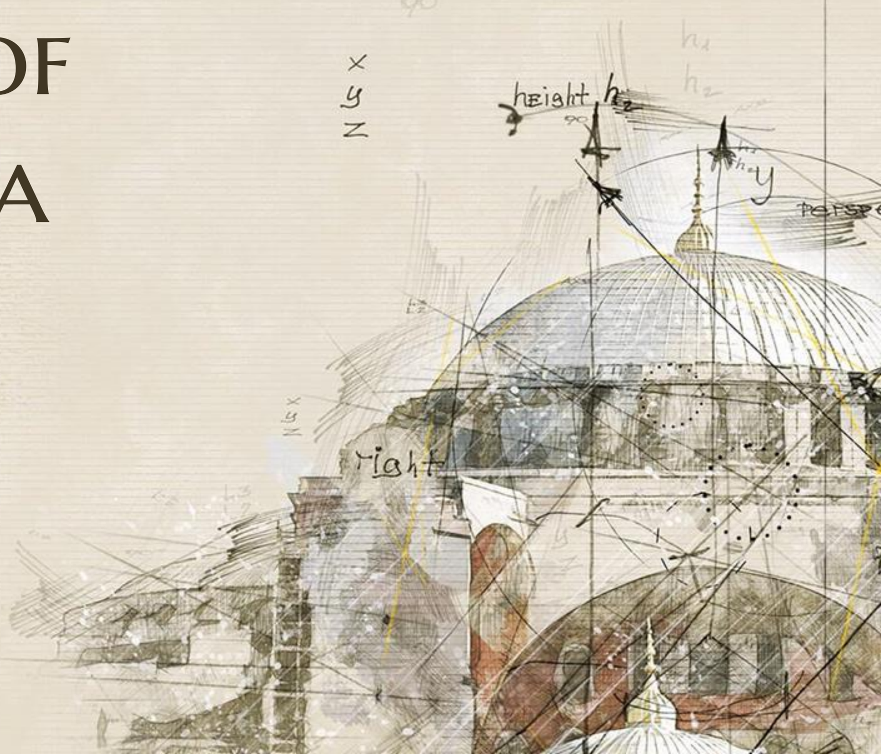




Whiting, Emily, John Ochsendorf, Frédo Durand, Emily Whiting, John Ochsendorf, and Frédo Durand. "Procedural Modeling of Structurally-Sound Masonry Buildings." In *ACM Transactions on Graphics (TOG)*, 28:112. ACM, 2009.

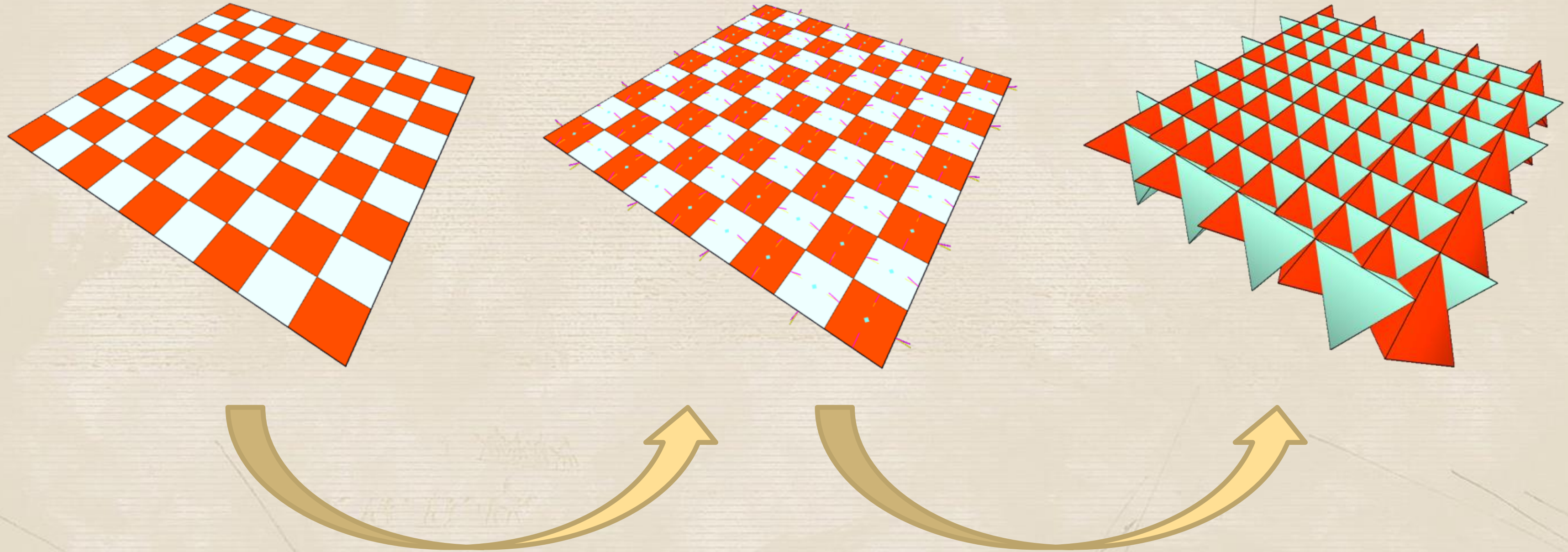
Even crabs can do it!

# 3.
# Assemblies made of Convex Polyhedra

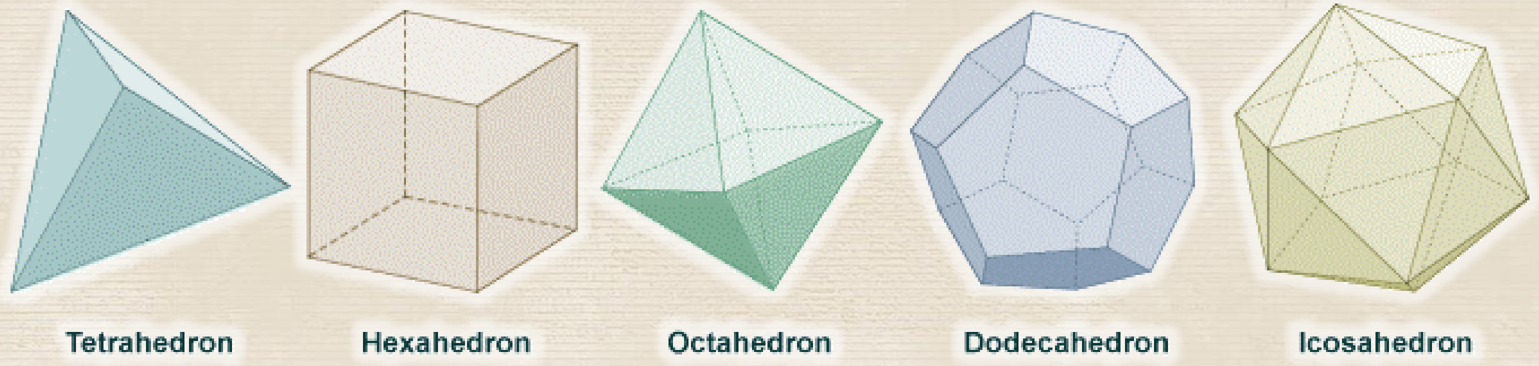
Structures made the hard way



# So, I work on TICs made of convex shapes



# Platonic Solids have convex interlocking properties

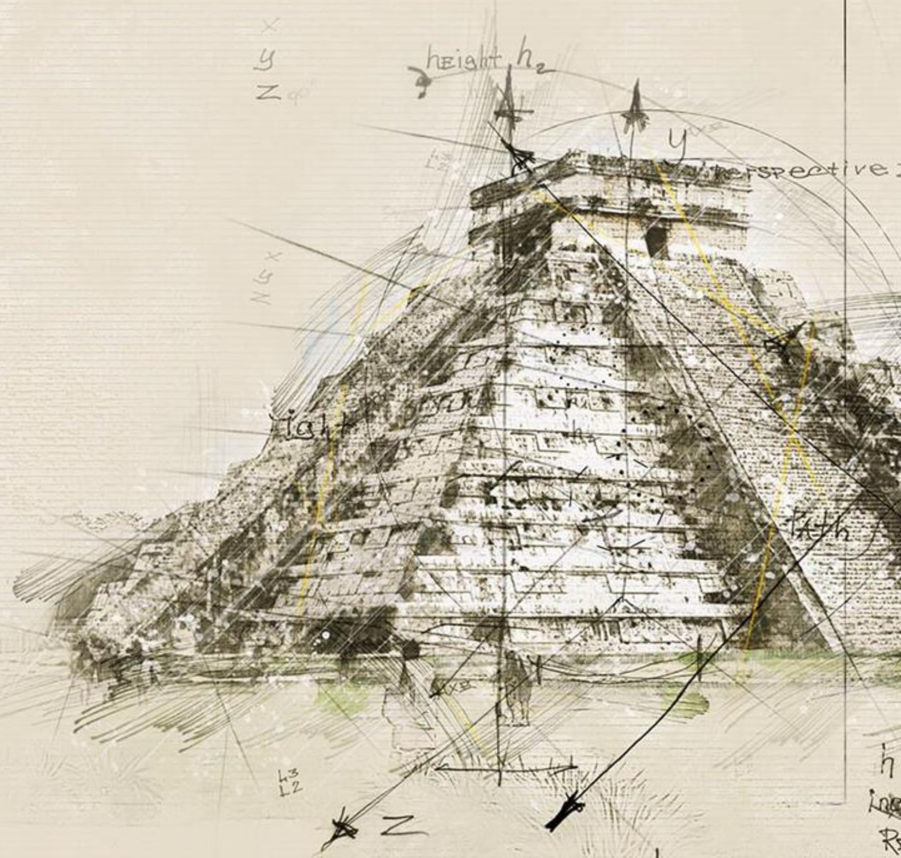





http://www.technologyuk.net/mathematics/geometry/platonic-solids.shtml

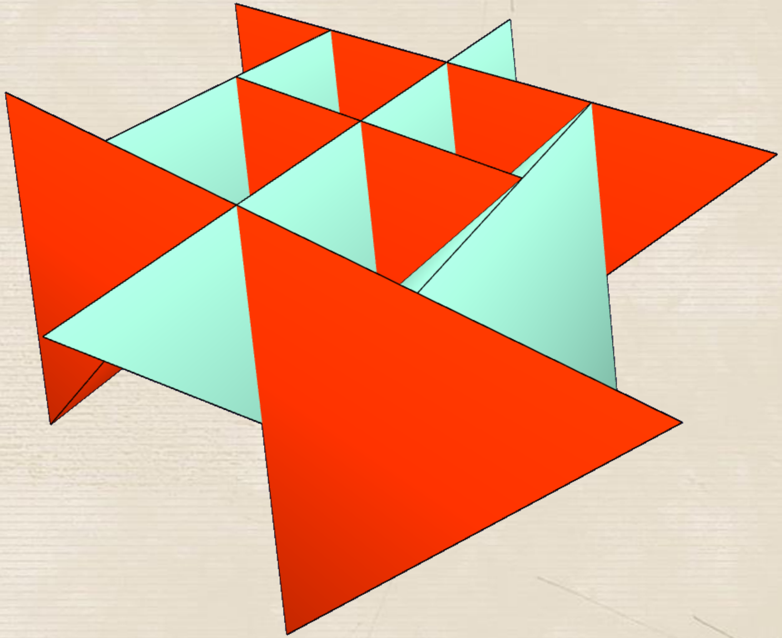
“ *It’s TIC time!*

*BE GENTLE WITH THOSE ANIMALS!*

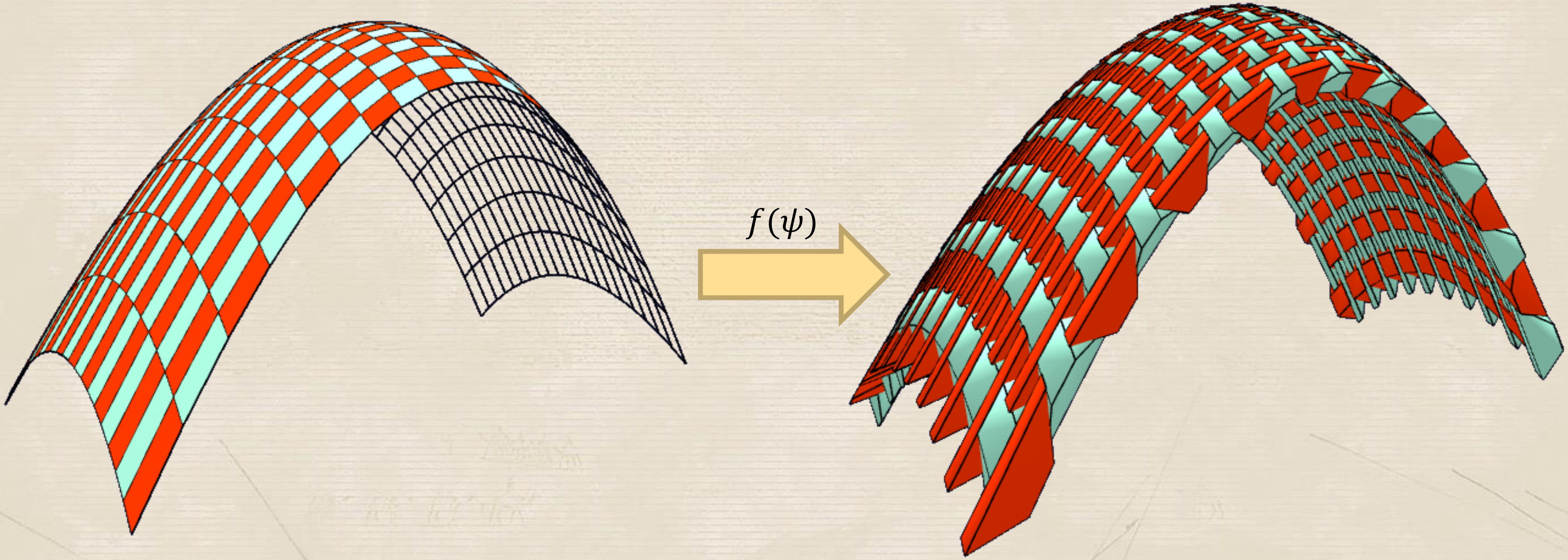


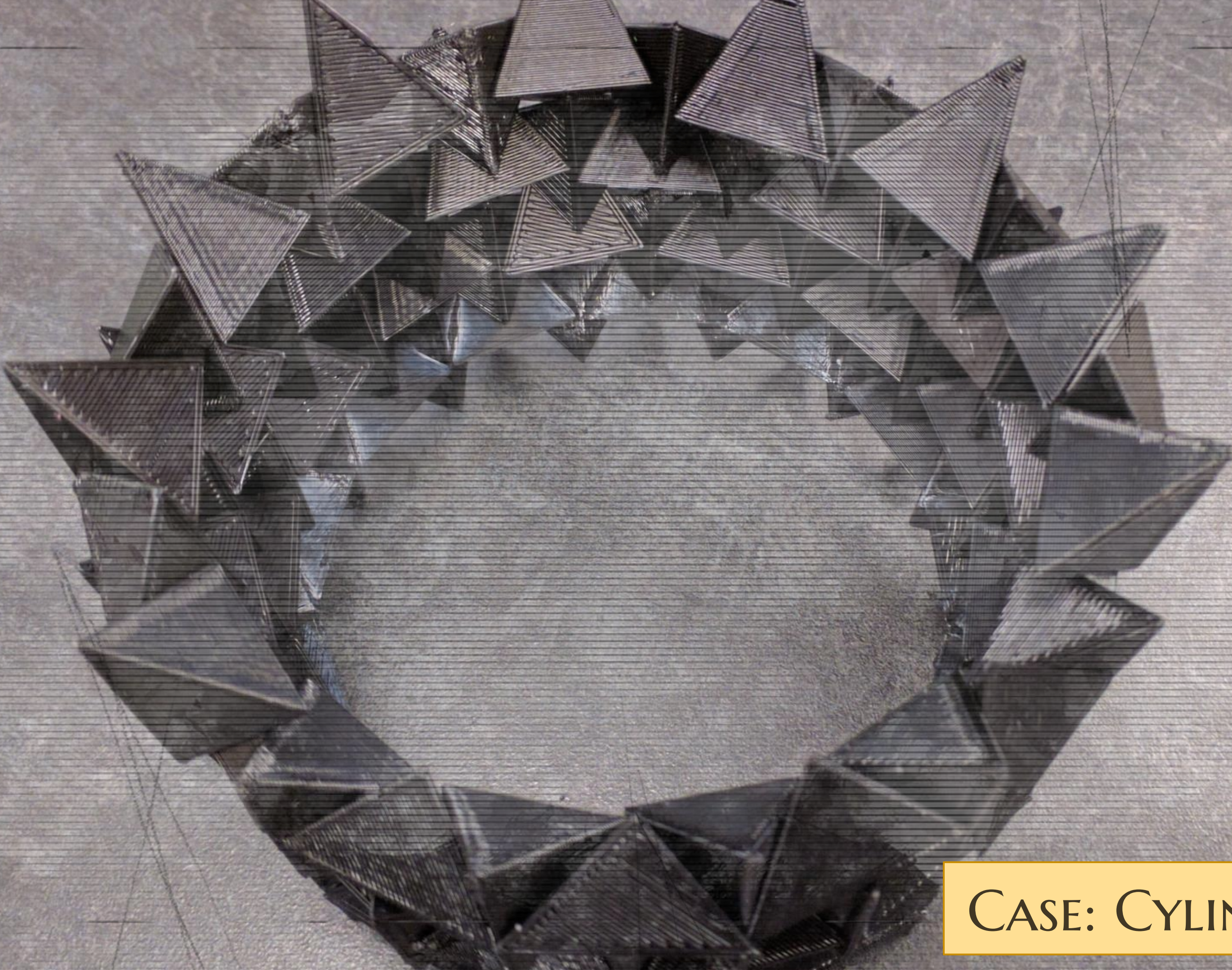
# I dare you!

Take 9 tetrahedra and put them together as shown. Then, lift them up and keep the piece in the middle on interlocking (5 seconds).

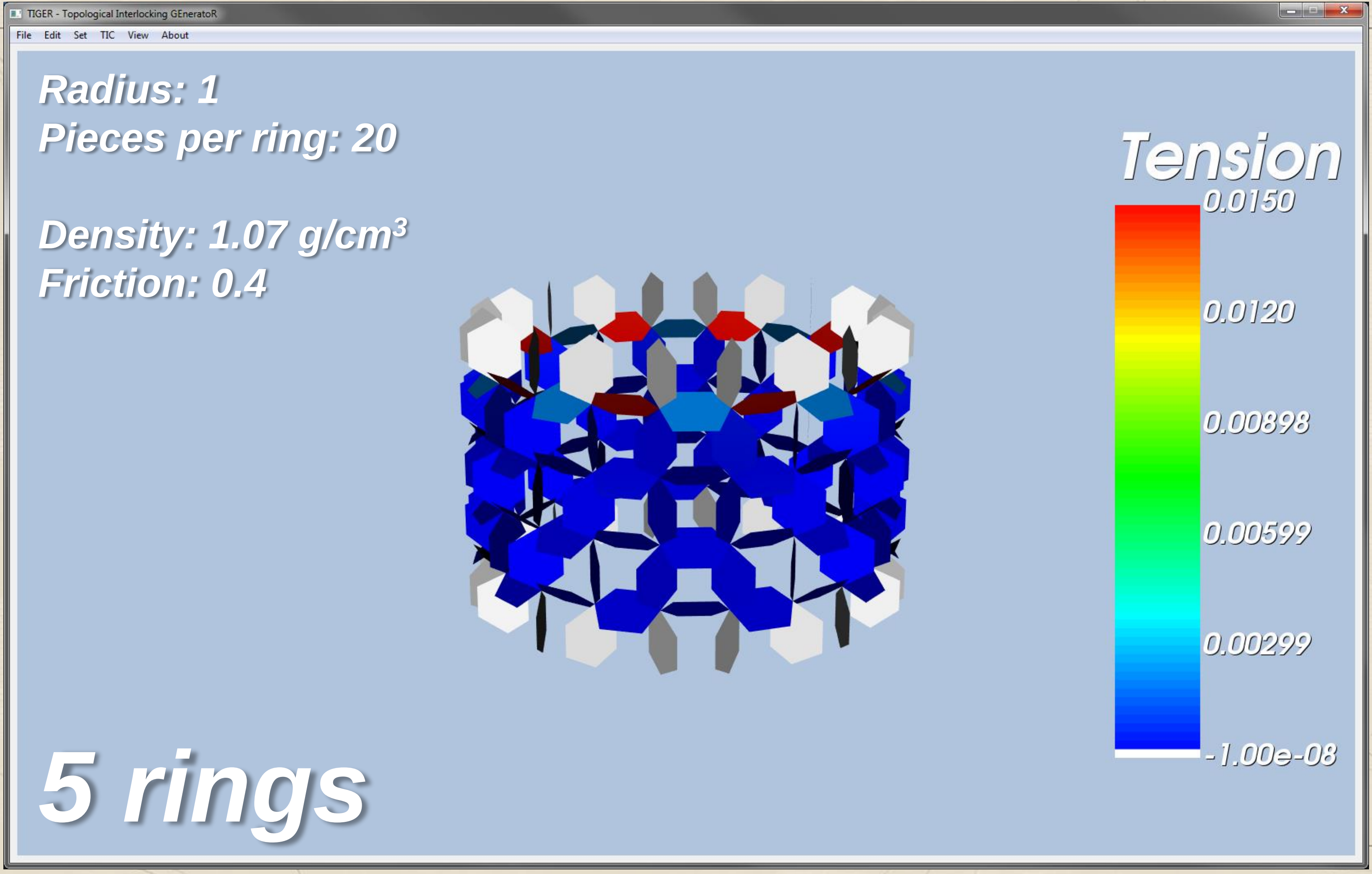


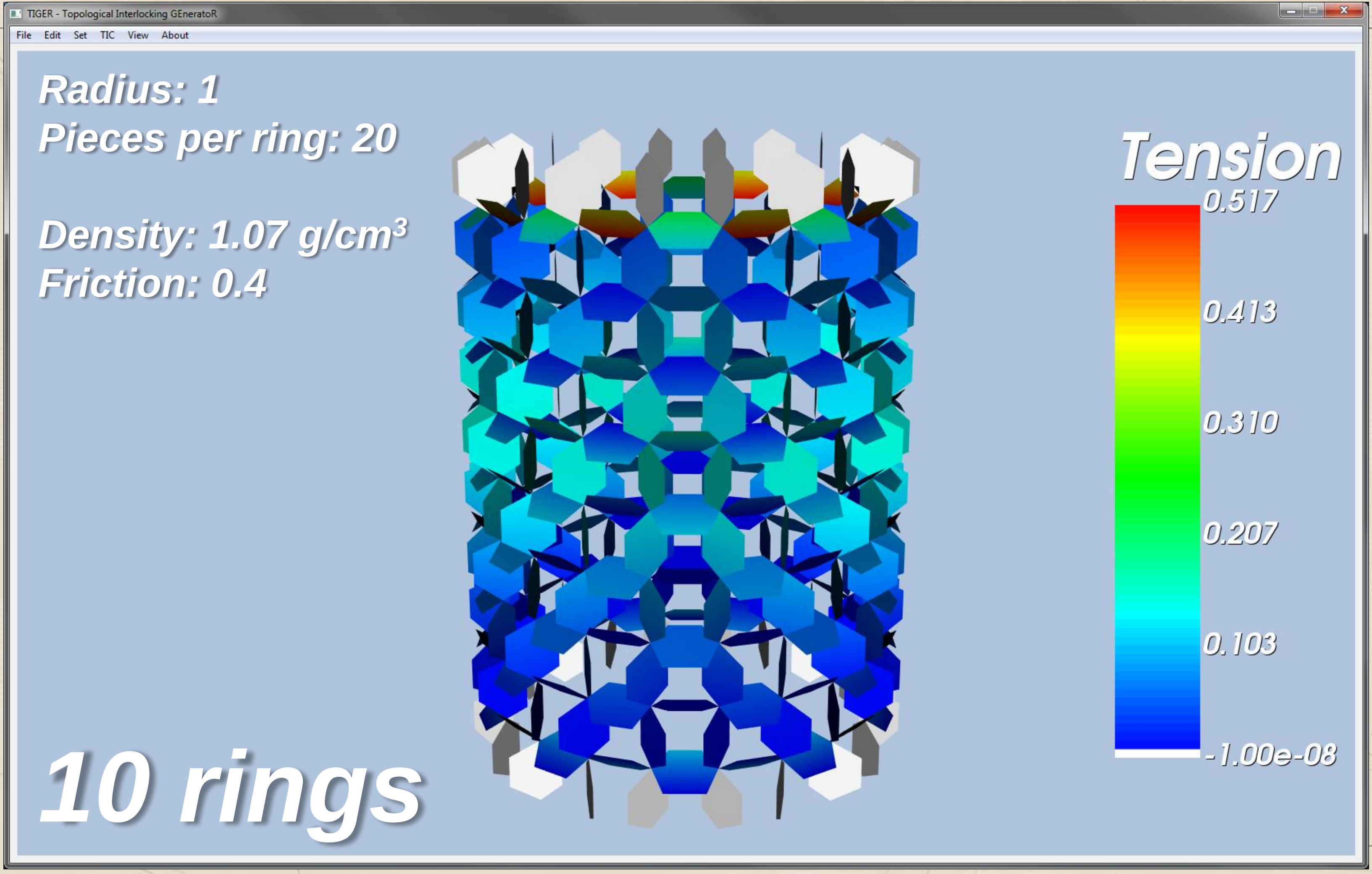
# BIG PROBLEM: FUNCTIONAL 3D TICs

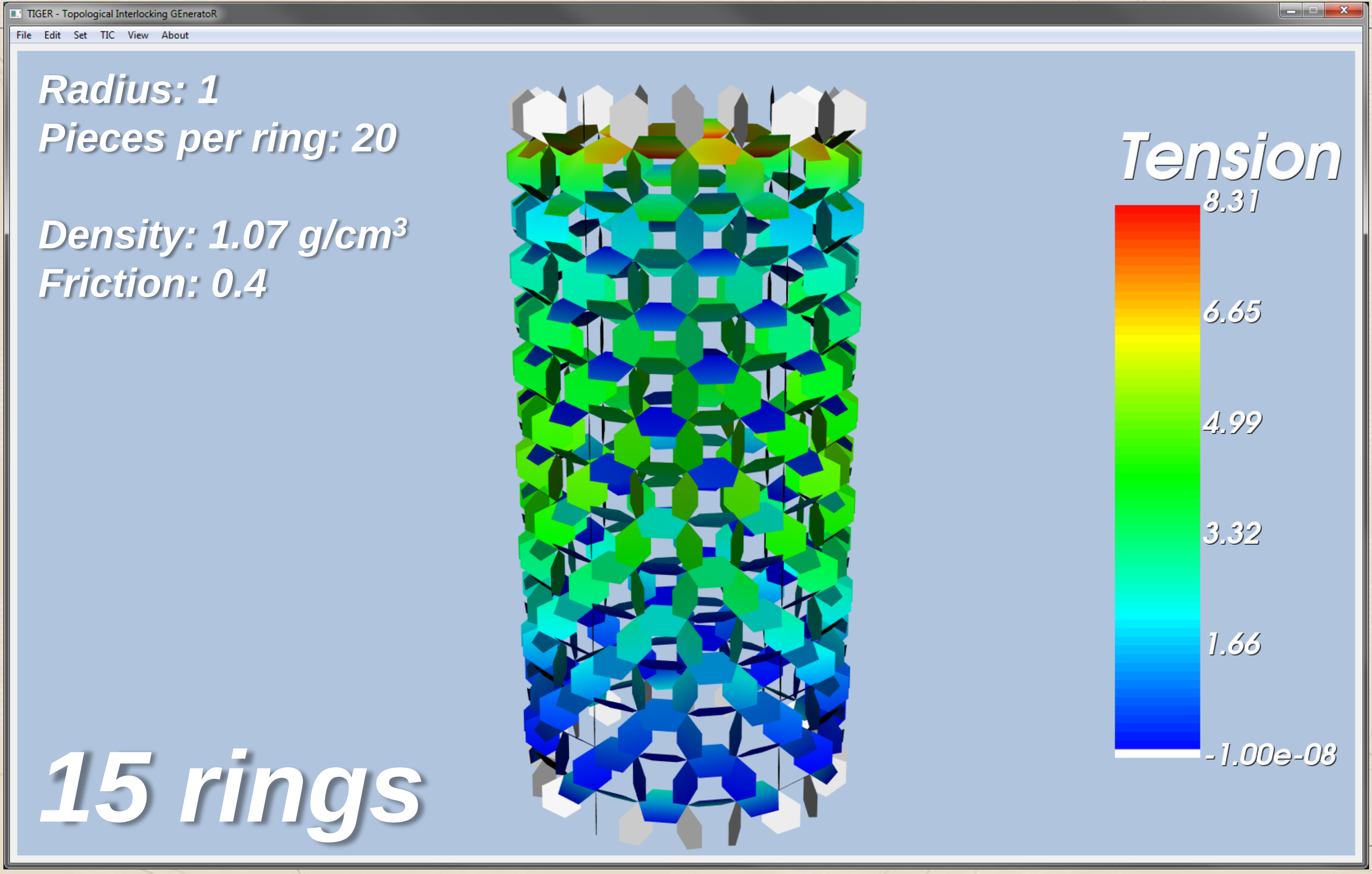


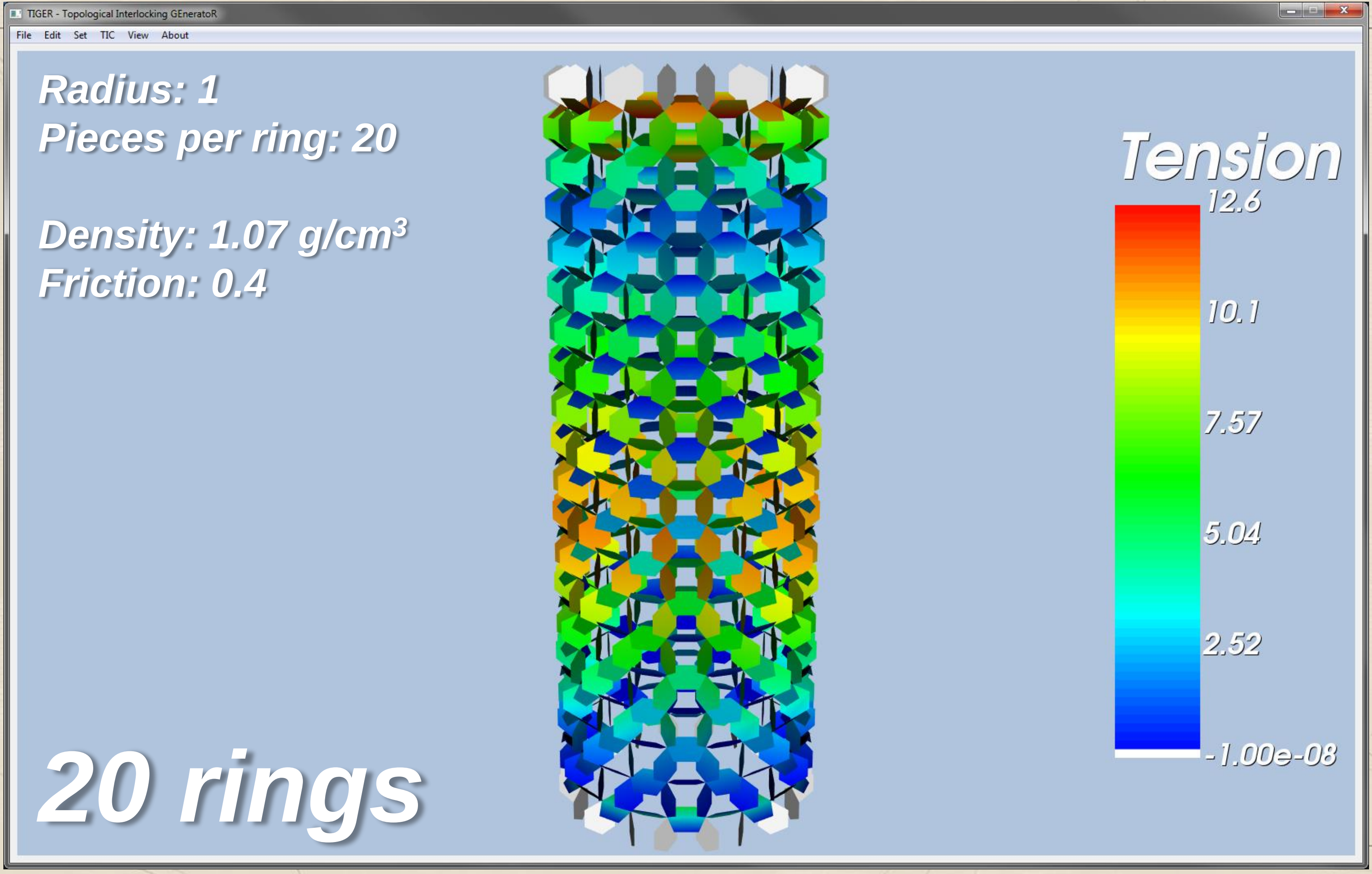


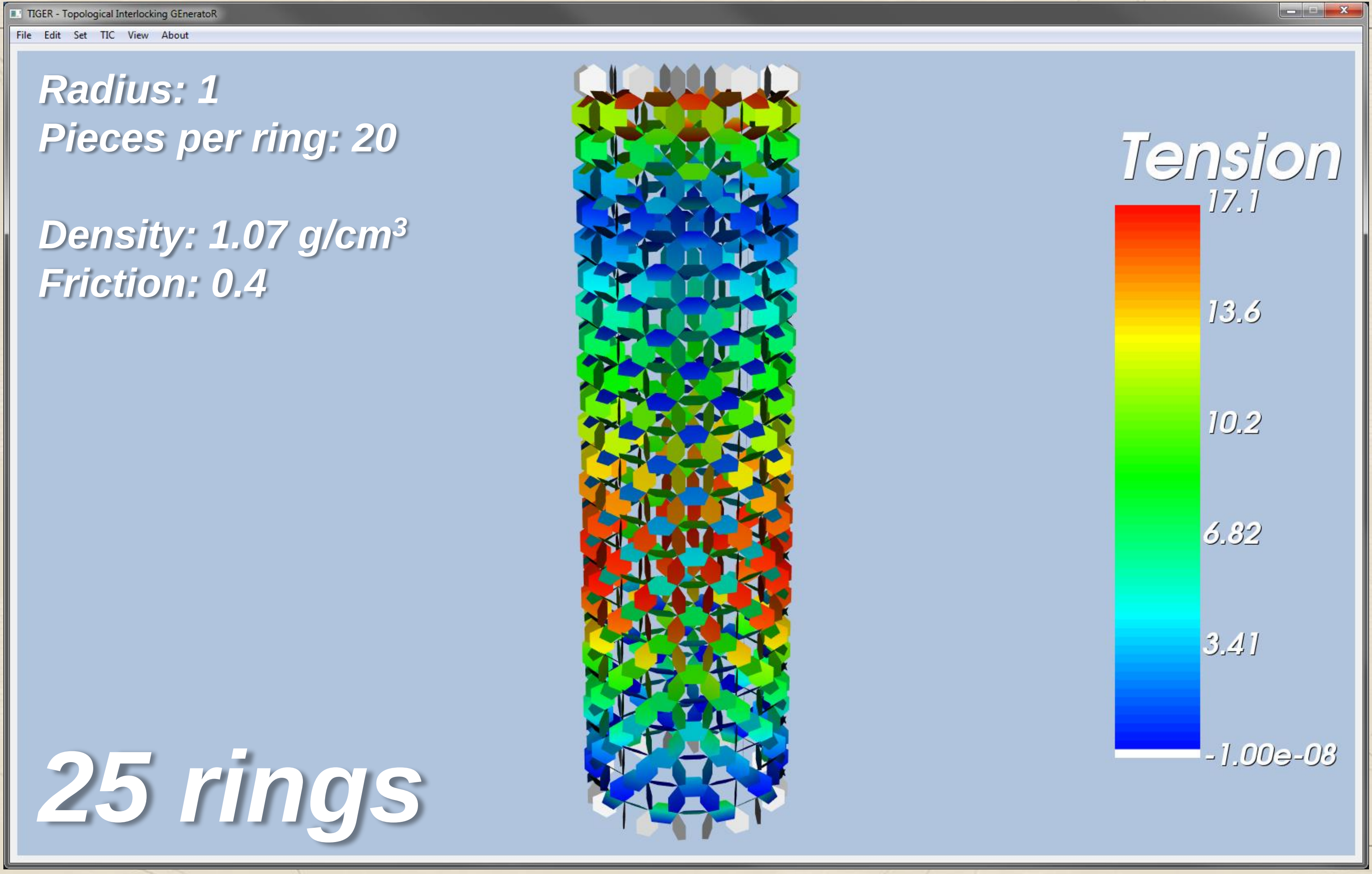
## Case: Cylindrical TIC

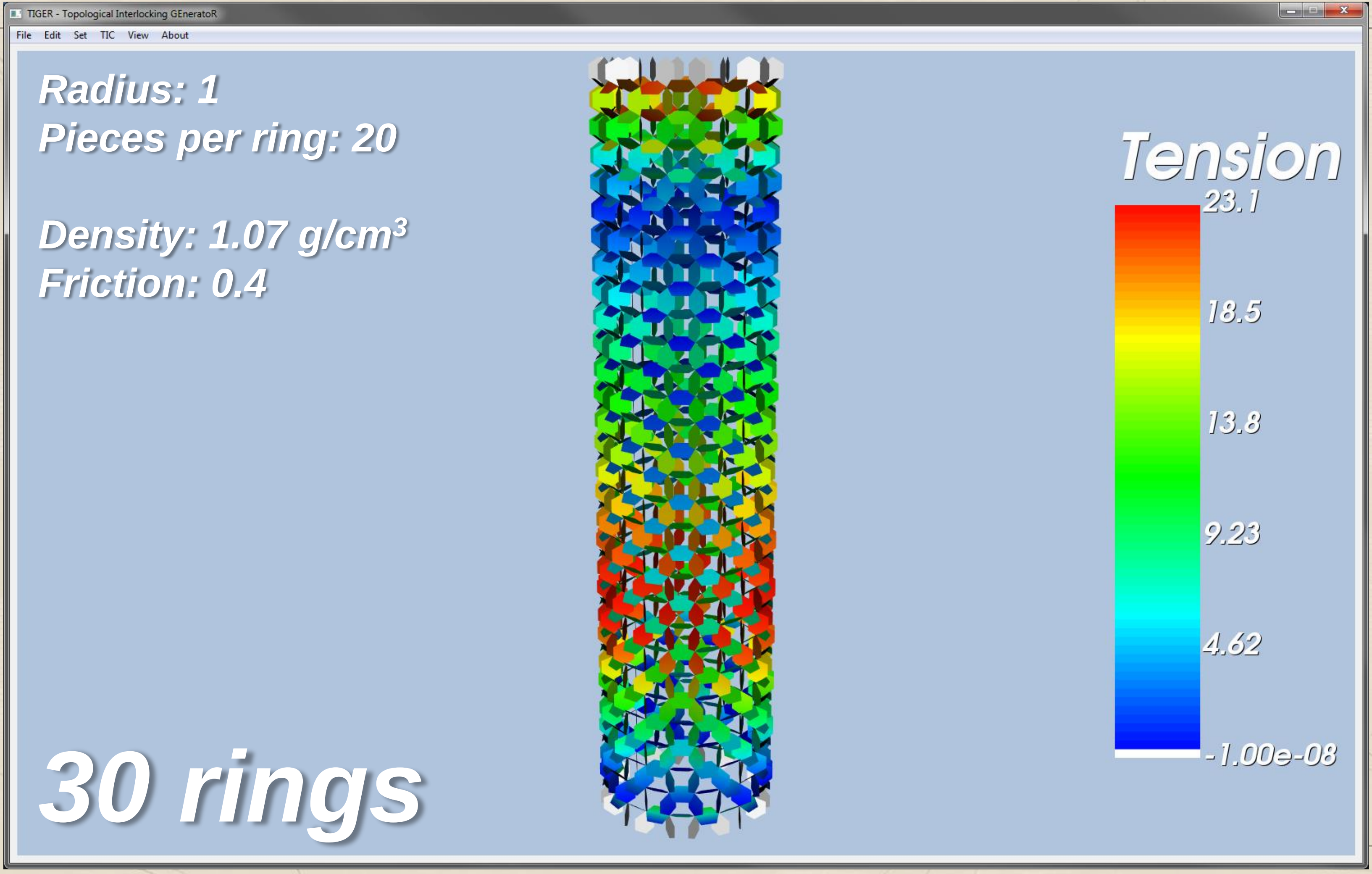

TIGER - Topological Interlocking GEneratoR
File Edit Set TIC View About
Radius: 1
Pieces per ring: 20
Density: 1.07 g/cm³
Friction: 0.4
Tension
0.0150
0.0120
0.00898
0.00599
0.00299
-1.00e-08
5 rings

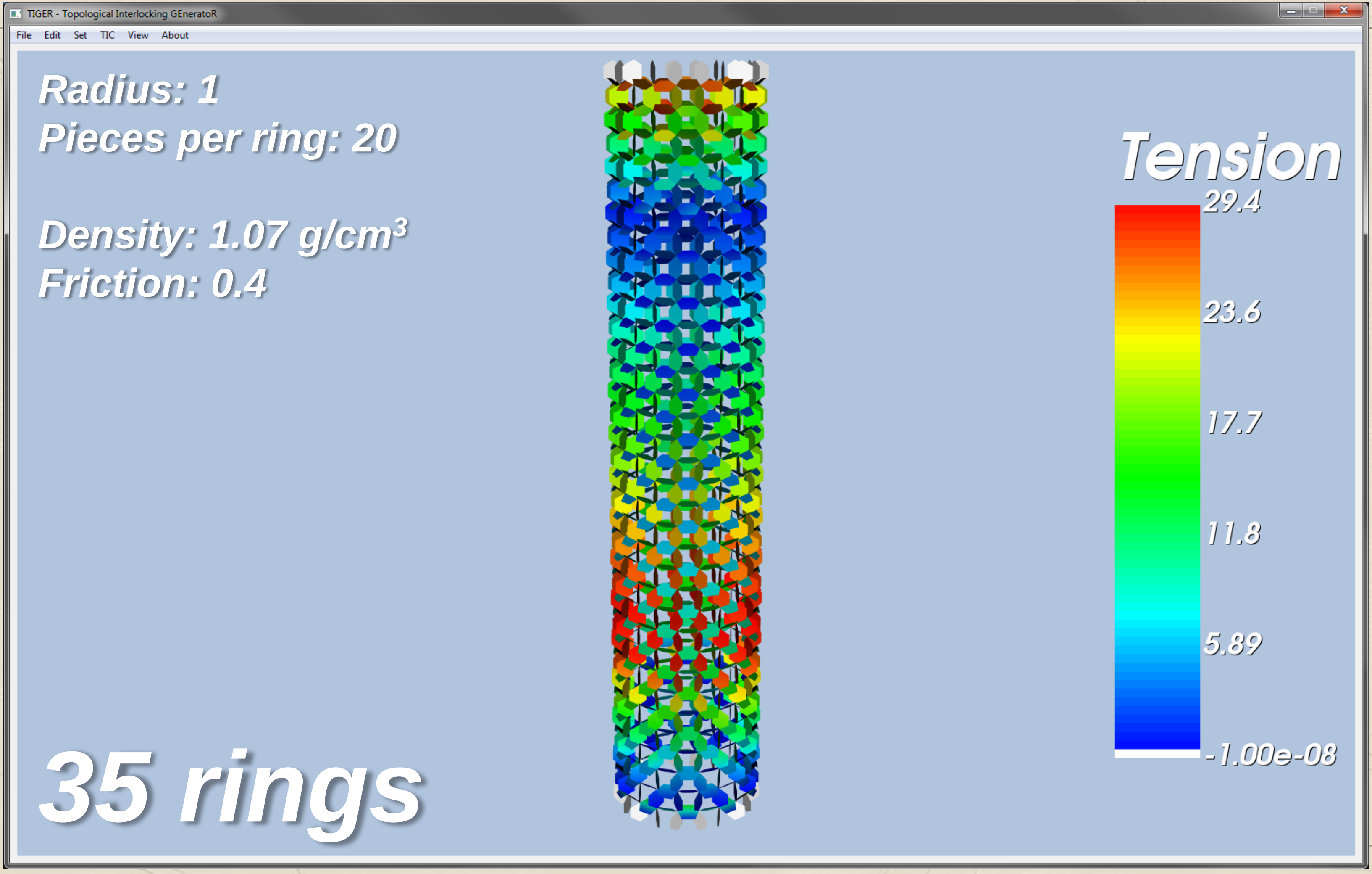

TIGER - Topological Interlocking GEneratoR
File Edit Set TIC View About
Radius: 1
Pieces per ring: 20
Density: 1.07 g/cm3
Friction: 0.4
10 rings
Tension
0.517
0.413
0.310
0.207
0.103
-1.00e-08

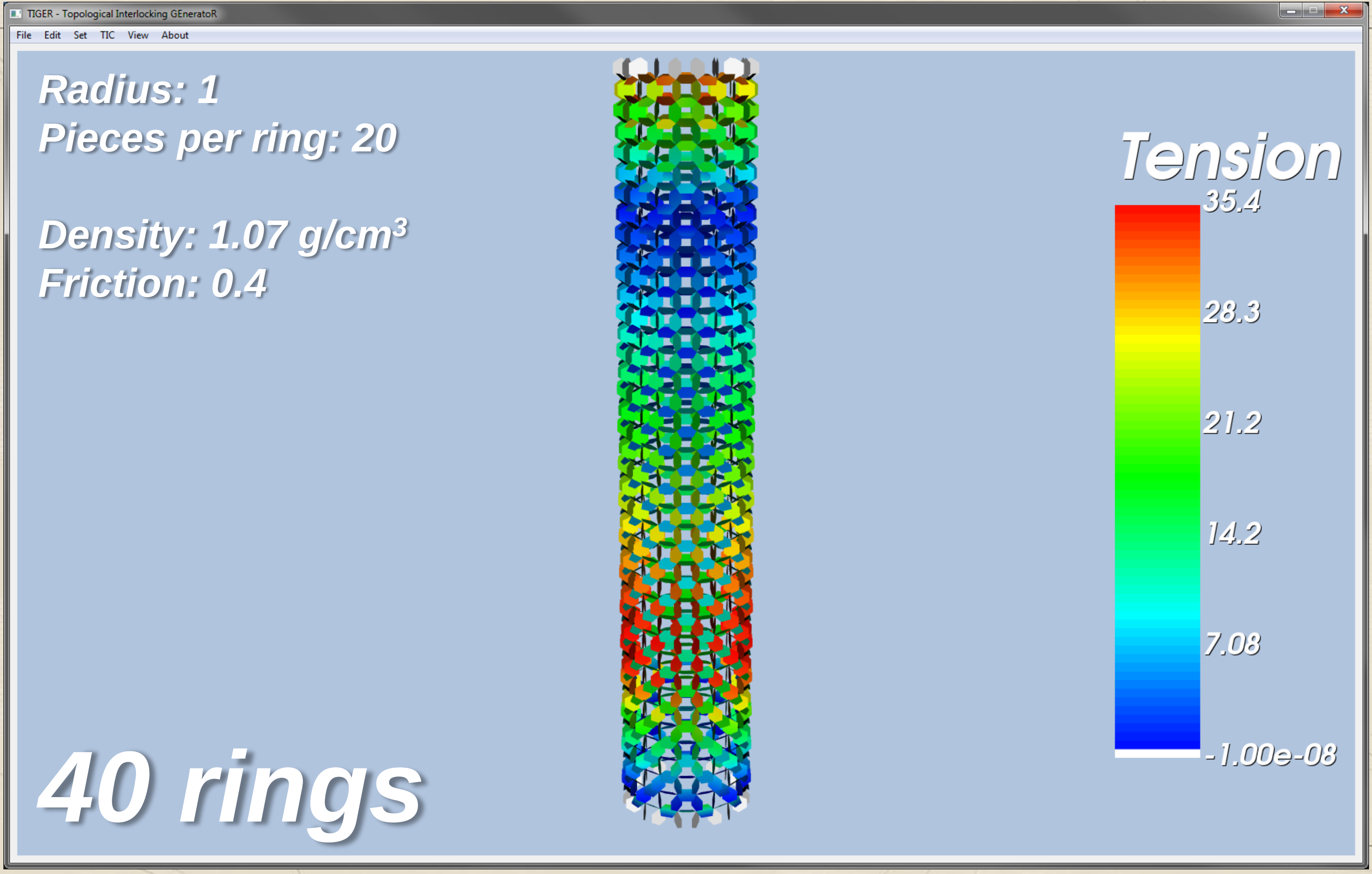

TIGER - Topological Interlocking GEneratoR
File Edit Set TIC View About
Radius: 1
Pieces per ring: 20
Density: 1.07 g/cm³
Friction: 0.4
15 rings
Tension
8.31
6.65
4.99
3.32
1.66
-1.00e-08

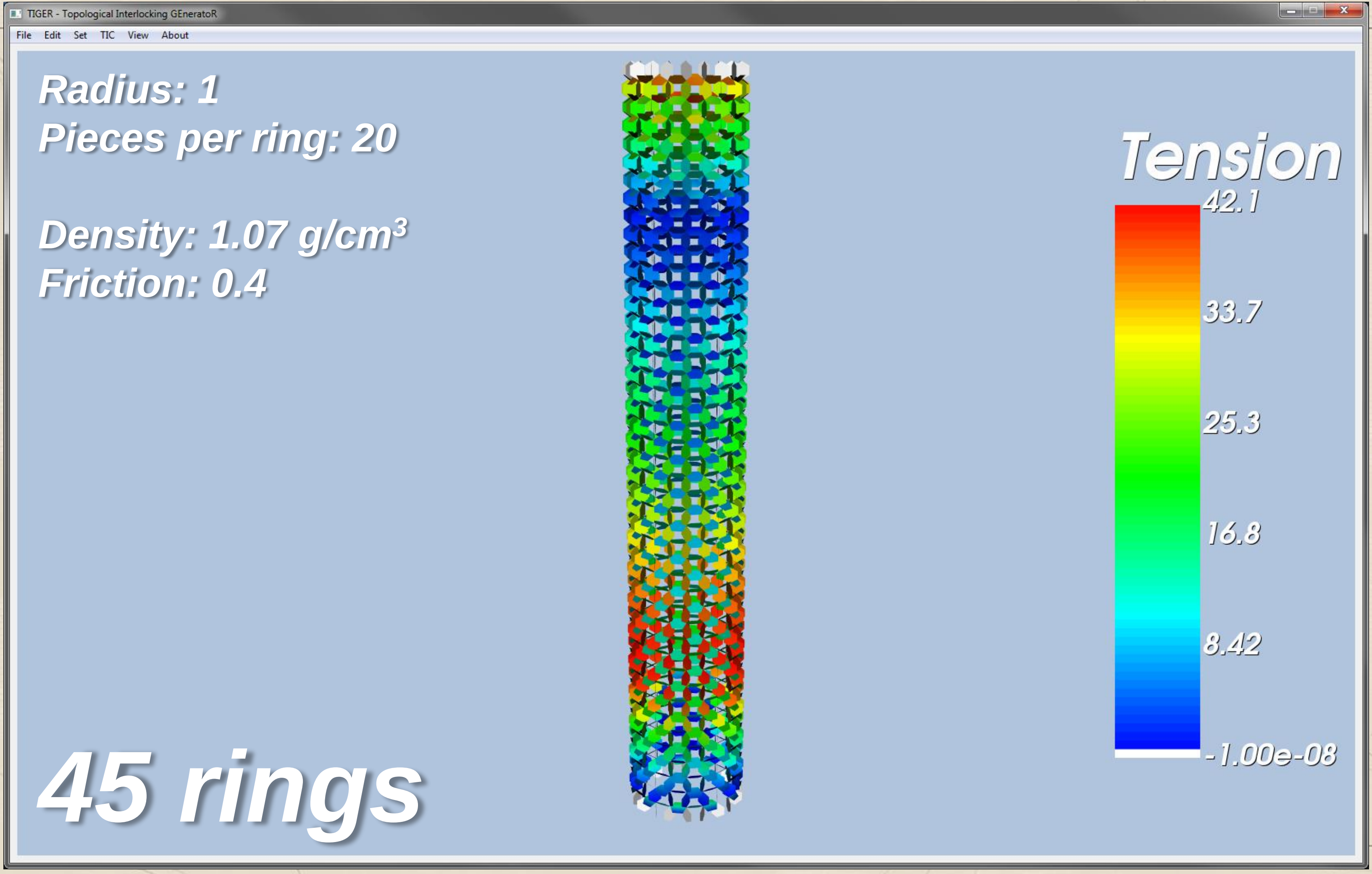

TIGER - Topological Interlocking GEneratoR
File Edit Set TIC View About
Radius: 1
Pieces per ring: 20
Density: 1.07 g/cm3
Friction: 0.4
20 rings
Tension
12.6
10.1
7.57
5.04
2.52
-1.00e-08

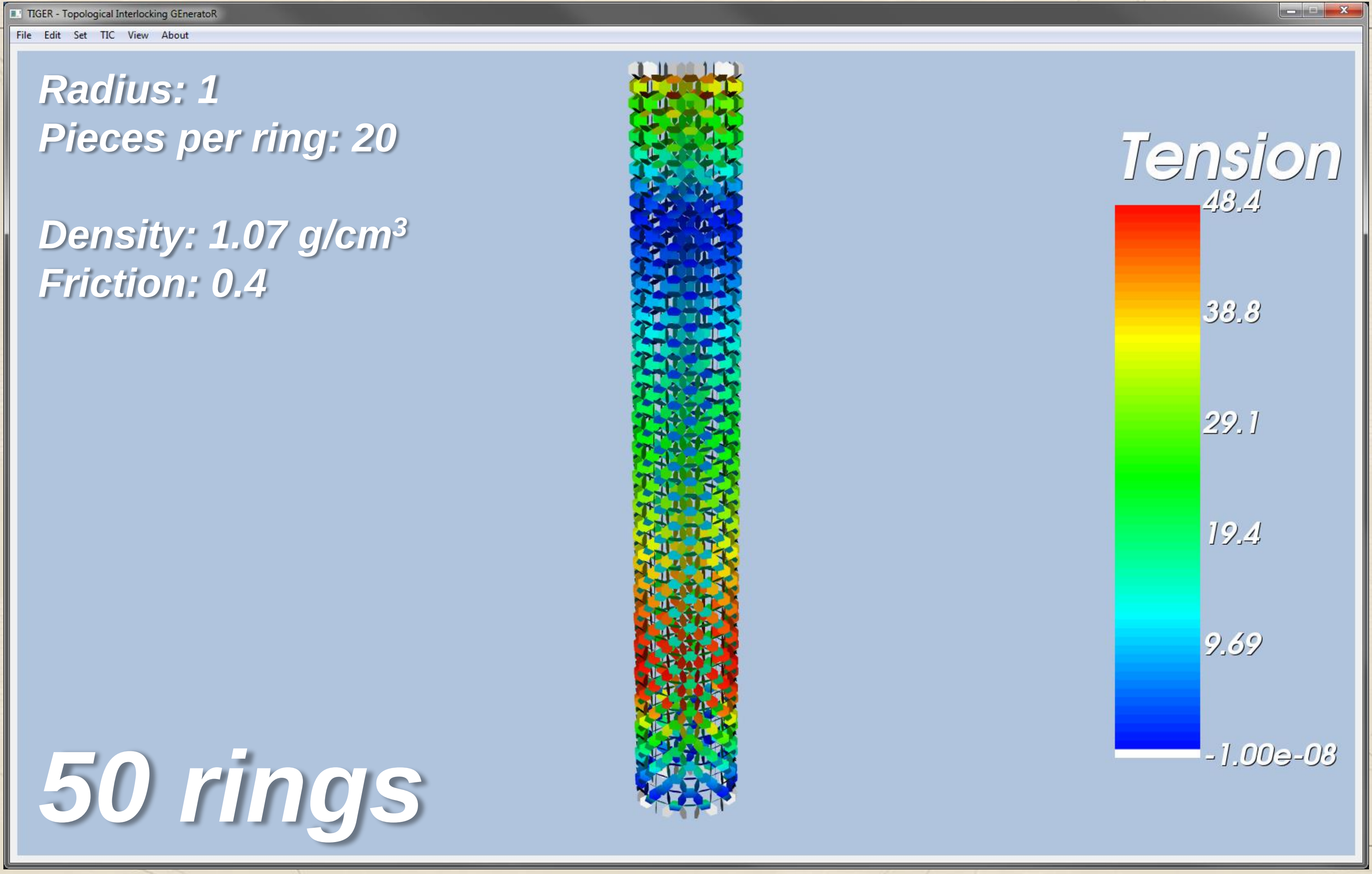

TIGER - Topological Interlocking GEneratoR
File Edit Set TIC View About
Radius: 1
Pieces per ring: 20
Density: 1.07 g/cm³
Friction: 0.4
25 rings
Tension
17.1
13.6
10.2
6.82
3.41
-1.00e-08

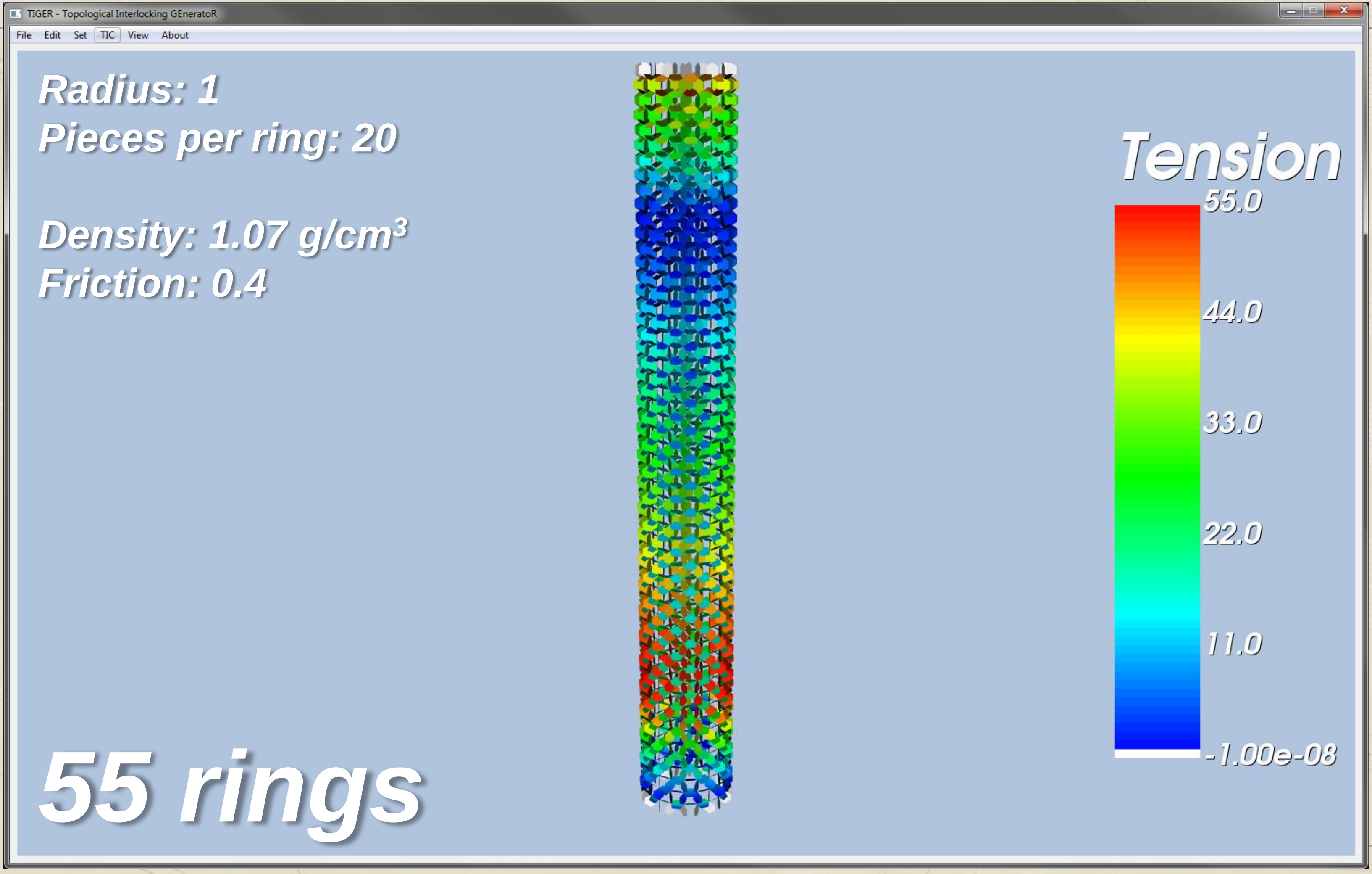

TIGER - Topological Interlocking GEneratoR
File Edit Set TIC View About
Radius: 1
Pieces per ring: 20
Density: 1.07 g/cm3
Friction: 0.4
30 rings
Tension
23.1
18.5
13.8
9.23
4.62
-1.00e-08

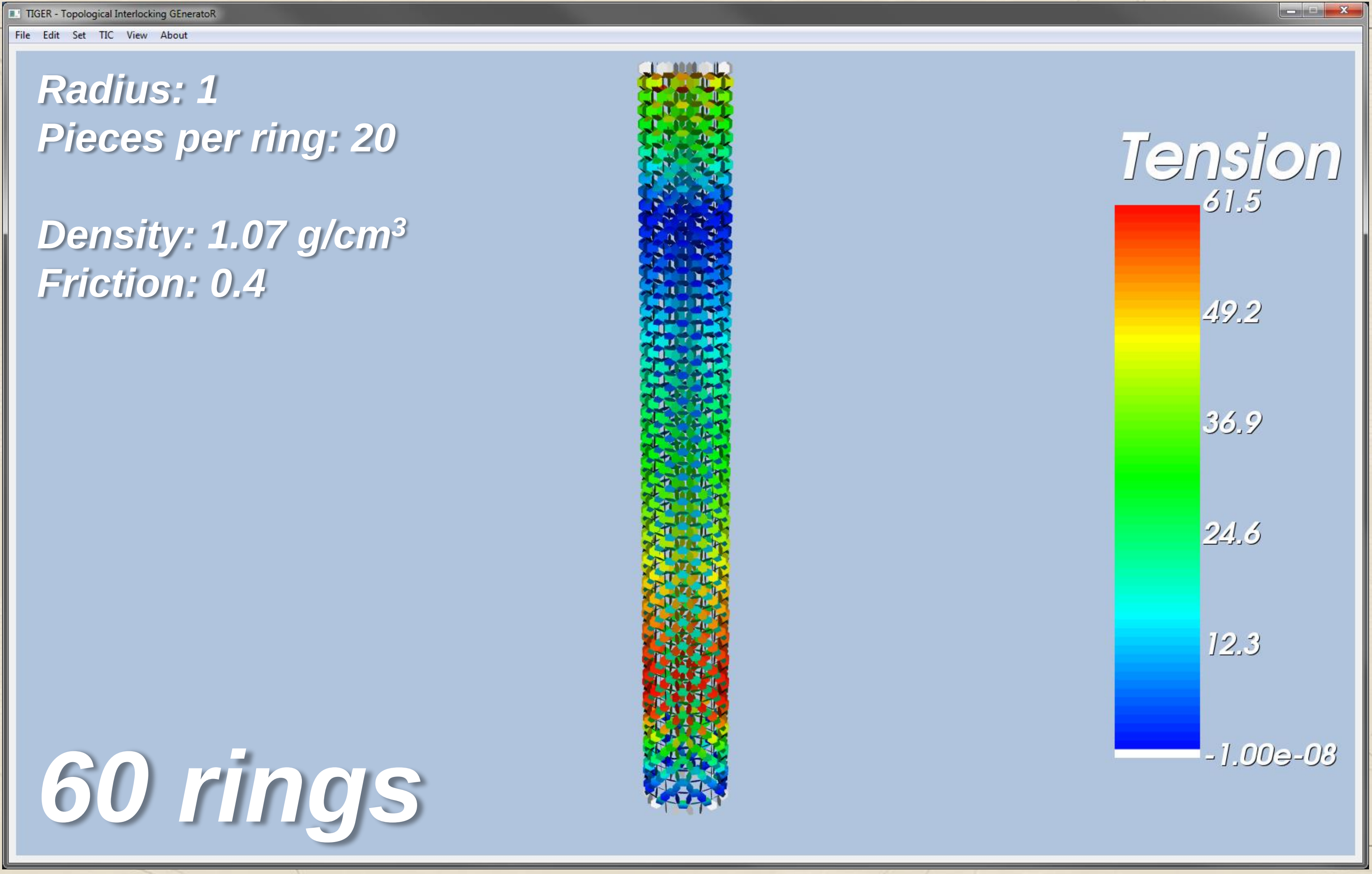

TIGER - Topological Interlocking GEneratoR
File Edit Set TIC View About
Radius: 1
Pieces per ring: 20
Density: 1.07 g/cm³
Friction: 0.4
35 rings
Tension
29.4
23.6
17.7
11.8
5.89
-1.00e-08

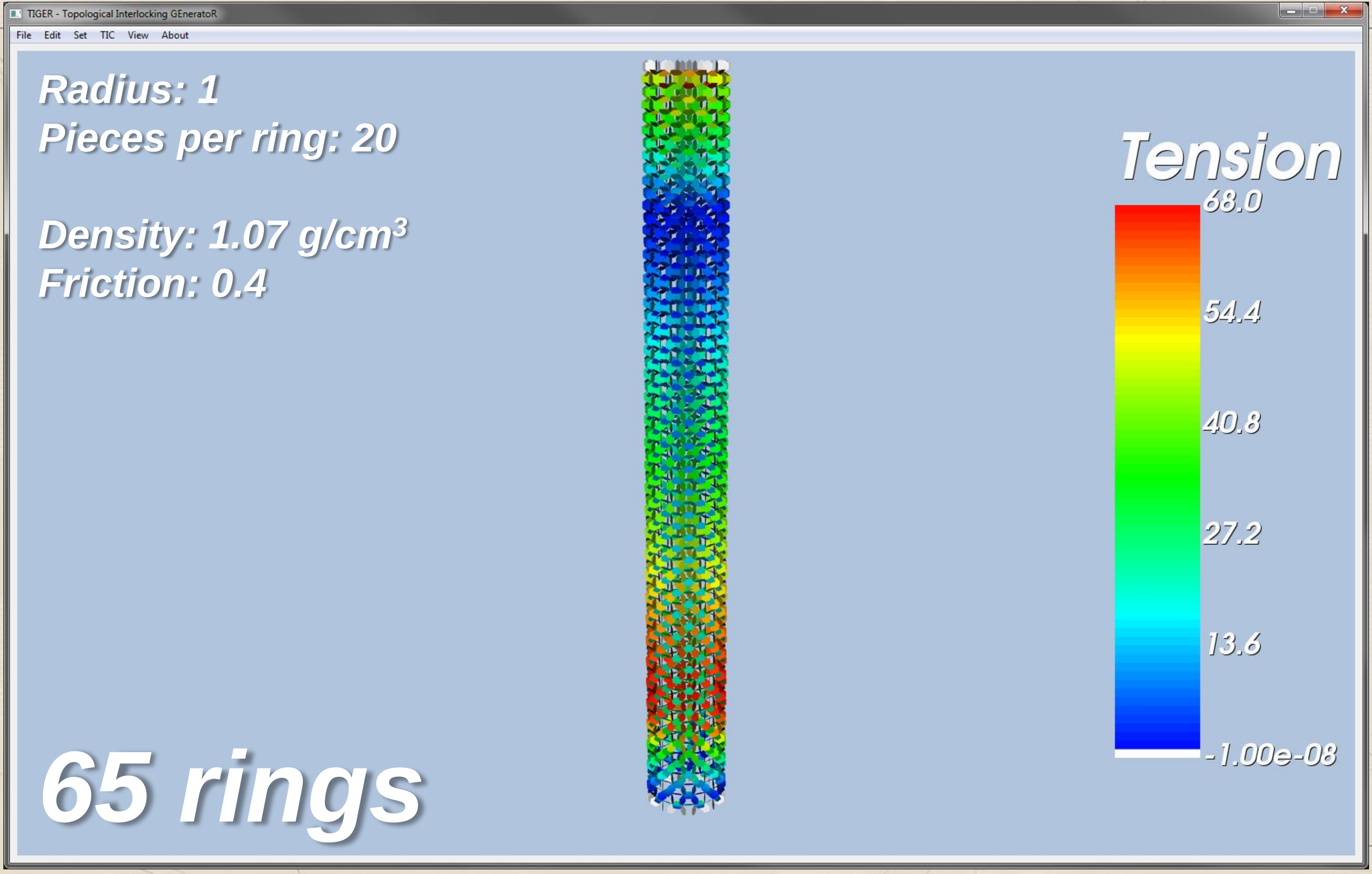

TIGER - Topological Interlocking GEneratoR
File Edit Set TIC View About
Radius: 1
Pieces per ring: 20
Density: 1.07 g/cm³
Friction: 0.4
40 rings
Tension
35.4
28.3
21.2
14.2
7.08
-1.00e-08

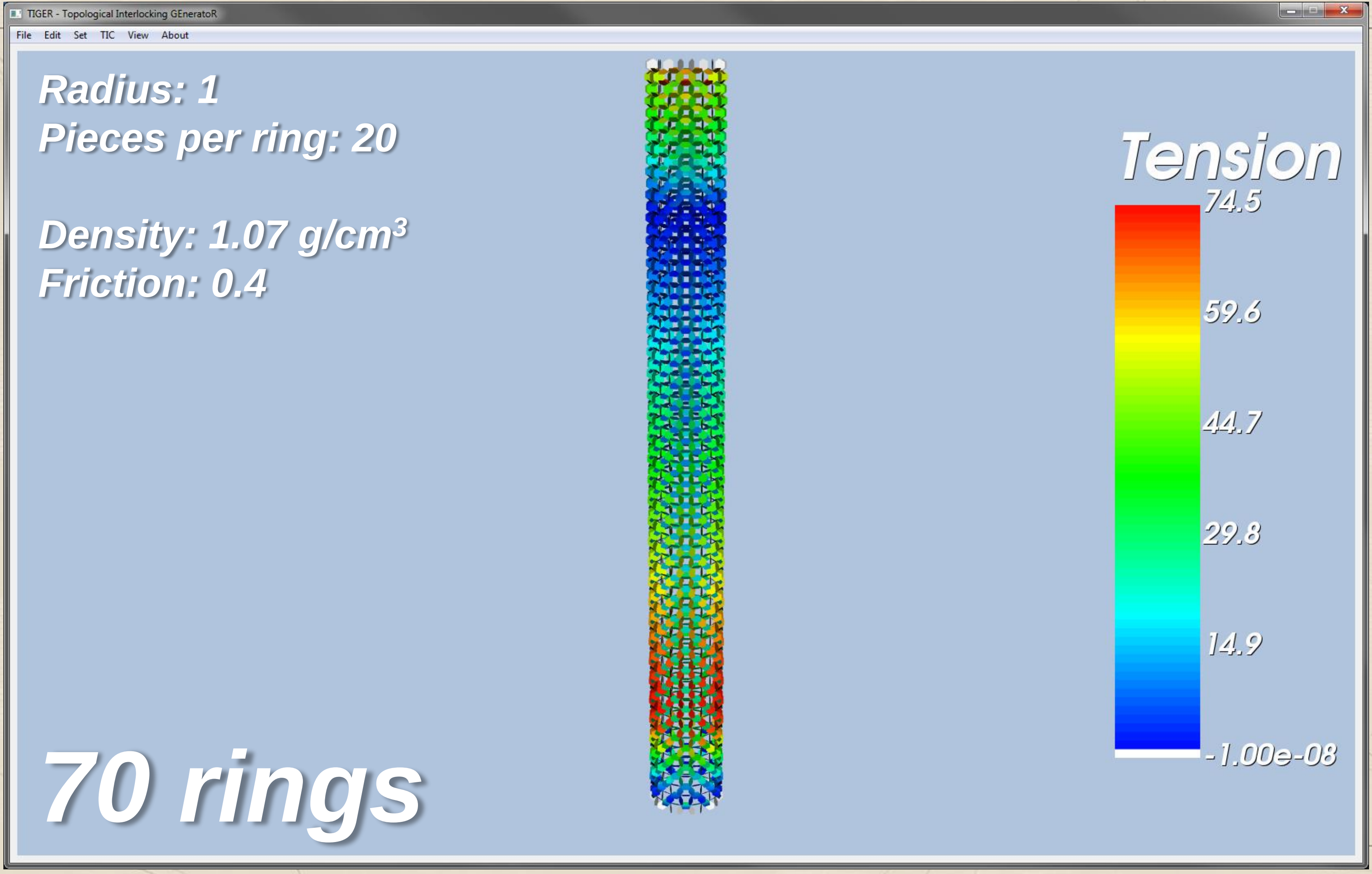

TIGER - Topological Interlocking GEneratoR
File Edit Set TIC View About
Radius: 1
Pieces per ring: 20
Density: 1.07 g/cm³
Friction: 0.4
45 rings
Tension
42.1
33.7
25.3
16.8
8.42
-1.00e-08


TIGER - Topological Interlocking GEneratoR
File Edit Set TIC View About
Radius: 1
Pieces per ring: 20
Density: 1.07 g/cm³
Friction: 0.4
50 rings
Tension
48.4
38.8
29.1
19.4
9.69
-1.00e-08


TIGER - Topological Interlocking GEneratoR
File Edit Set TIC View About
Radius: 1
Pieces per ring: 20
Density: 1.07 g/cm3
Friction: 0.4
55 rings
Tension
55.0
44.0
33.0
22.0
11.0
-1.00e-08

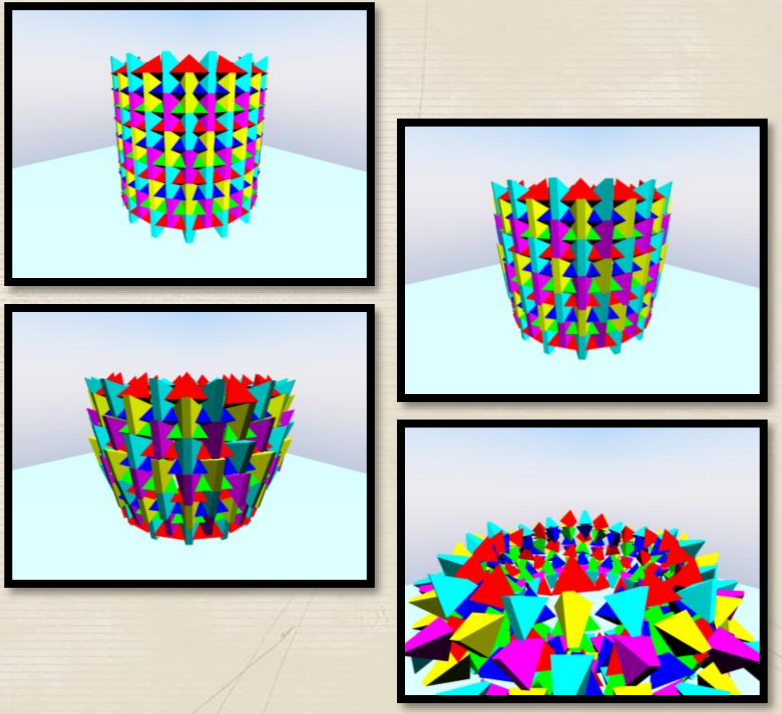

TIGER - Topological Interlocking GEneratoR
File Edit Set TIC View About
Radius: 1
Pieces per ring: 20
Density: 1.07 g/cm3
Friction: 0.4
60 rings
Tension
61.5
49.2
36.9
24.6
12.3
-1.00e-08


TIGER - Topological Interlocking GEneratoR
File Edit Set TIC View About
Radius: 1
Pieces per ring: 20
Density: 1.07 g/cm³
Friction: 0.4
65 rings
Tension
68.0
54.4
40.8
27.2
13.6
-1.00e-08


TIGER - Topological Interlocking GEneratoR
File Edit Set TIC View About
Radius: 1
Pieces per ring: 20
Density: 1.07 g/cm³
Friction: 0.4
70 rings
Tension
74.5
59.6
44.7
29.8
14.9
-1.00e-08

# What's wrong with simulations?

- Yes/no answer
- What should we simulate?
- Makes sense?



# Questions

- How do we generate pieces different than antiprisms?
- How about Archimedean and Catalan Solids?
- Do TICs guarantee stability?
- Does the piece shape affect stability?
- Does it work using free form geometric domains (e.g., Stanford Bunny)?

# Episode 2

## General Mid-Section Evolution for TIC Generation

November 20th, 2019

# Thanks!

## Any questions?

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- Office #19

# Credits

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